SOME APPLICATIONS OF A THEOREM OF CHEVALLEY

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1. **Introduction.** Chevalley [3] has proved the following theorem:

Let $F_1, \cdots, F_k$ be polynomials in the indeterminates $x_1, \cdots, x_n$, with coefficients in $GF(q)$, which vanish at $(0, \cdots, 0)$ and such that the sum of the degrees $<n$. Then the F_i vanish in at least one additional point $(a_1, \cdots, a_n)$, $a_i \varepsilon GF(q)$.

In the present note we make several applications of this theorem. In the first place we show that certain systems of equations with polynomial coefficients have non-trivial solutions. For example if $f(u_1, \cdots, u_s)$ is a polynomial of degree $\leq k$ and with coefficients in $GF[q, x]$, $f(0, \cdots, 0) = 0$, then the equation

$$f(U_1, \cdots, U_s) = 0 \qquad (U_i \varepsilon GF[q, x]) \tag{1.1}$$

always has a non-trivial solution in polynomials U_i provided $s \geq k^2 + 1$. Moreover, this value of s cannot in general be diminished; that is, there exist equations (1.1) of degree k which have only the trivial solution for $s = k^2$.

In the second place we apply the theorem to certain problems of approximation in the field $GF\{q, x\}$ which consists of the numbers

$$\sum_{-\infty}^{m} c_i x^i \qquad (c_i \varepsilon GF(q)). \tag{1.2}$$

The following two results may be quoted:

(i) If α is a number of the form (1.2) and $m \geq 1$, $k \geq 1$, there exist polynomials $A, B \varepsilon GF[q, x]$, $A \neq 0$, $\deg A \leq km$, such that

$$\deg (A^k\alpha - B) < -m. \tag{1.3}$$

This result is the analog of a theorem proved by Vinogradoff and improved by Heilbronn [6]. For an extension of (1.3), see Theorems 4 and 5 below.

(ii) Let $f(u_1, \cdots, u_s)$ be a polynomial of degree $\leq k$ and without constant term; let the coefficients of f belong to $GF\{q, x\}$. Then provided $s \geq k^2 + 1$ there exist polynomials $U_1, \cdots, U_s \varepsilon GF[q, x]$ not all 0 and such that $\deg f(U_1, \cdots, U_s) < -r$, where r is preassigned (§7). This result may be compared with a theorem proved by Davenport and Heilbronn on indefinite quadratic forms [4].

Warning [8] has given a lower bound for the number of solutions in Chevalley's theorem. This result applied to some of the problems discussed in this paper does not, however, lead to the correct order of magnitude for the number of solutions. In certain cases more precise results can be obtained using other methods; these questions will be treated in other papers.

For other results on null forms see Albert [1].

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