PRIME RINGS

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This paper is the beginning of a projected study of the structure of prime rings, that is, of rings in which the zero ideal is prime. Fundamental in this study is the concept of a prime right ideal. A right ideal I of a ring R is called prime if $\mathfrak{a}\mathfrak{b} \subseteq I$ implies that $\mathfrak{a} \subseteq I$, $\mathfrak{a}$ and $\mathfrak{b}$ right ideals of R with $\mathfrak{b} \neq 0$. For every right ideal I of the ring R there is a unique minimal prime right ideal $p(I)$ containing I. The mapping $I \to p(I)$ is a closure operation [9; 494] on the lattice of right ideals of R.

Let us denote by $\mathfrak{A}$ the set of all right ideals of R and by $\mathfrak{P}$ the set of all prime right ideals of R. It is assumed in the present paper that there exists a mapping $I \to I^*$ of $\mathfrak{A}$ onto a subset $\mathfrak{N}$ of $\mathfrak{P}$ having the following seven properties.

(P1) $$I^* \supseteq I.$$

(P2) $$I^{**} = I^*.$$

(P3) $$\text{If } I \supseteq I', \text{ then } I^* \supseteq I'^*.$$

(P4) $$0^* = 0.$$

(P5) $$\text{If } I \cap I' = 0, \text{ then } I^* \cap I'^* = 0.$$

(P6) $$aI^* \subseteq (aI)^*.$$

(P7) *$\mathfrak{N}$ has minimal non-zero elements.*

From (P1)–(P3), we see that $I \to I^*$ is a closure operation on $\mathfrak{A}$. If we let $I^* = p(I)$, then the ring R of all $n \times n$ matrices over the integers is an example of a ring with properties (P1)–(P7).

Any admissible right R-module M has a closure operation $N \to N^*$ induced by $\mathfrak{N}$ on the submodules of M. The main result of the paper is that $\mathfrak{N}$, the lattice of closed submodules of M, is isomorphic to the lattice of principal right ideals of a certain regular ring, the so-called extended centralizer of R over M [5]. This implies that R itself has a regular quotient ring E such that $\mathfrak{N}$ is isomorphic to the lattice of principal right ideals of E.

1. **Prime right ideals.** An ideal S of a (non-zero) ring R is called *prime* [7] if $\mathfrak{a}\mathfrak{b} \subseteq S$ implies that $\mathfrak{a} \subseteq S$ or $\mathfrak{b} \subseteq S$, $\mathfrak{a}$ and $\mathfrak{b}$ ideals (or r-ideals; or l-ideals) of R. The ring R itself is called a *prime ring* [7; 830] if 0 is a prime ideal of R.

A right ideal I of R will be called a *prime right ideal* of R if and only if $\mathfrak{a}\mathfrak{b} \subseteq I$ implies that $\mathfrak{a} \subseteq I$, $\mathfrak{a}$ and $\mathfrak{b}$ r-ideals of R with $\mathfrak{b} \neq 0$. Left primeness can be defined analogously.

It is evident that no ideal $S \neq 0$ of R can be contained in a prime right ideal I different from R. For $RS \subseteq S \subseteq I$ implies that $R \subseteq I$. Hence the concept

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