THE BASIS IN BANACH SPACE

By Albert Wilansky

The purpose of this paper is to supplement the existing literature on the basis in several ways. Along with new results, connections are shown between results of recent papers of James [4], Karlin [5], and Gelbaum [3]; also, it is shown that the lemma of James can be deduced from elementary considerations, in particular, without reference to the theorem of Eberlein on weak compactness. In fact, the methods yield this result for the special case of a space with a basis. The paper is self-contained except for references to [1].

We deal with a Banach space. Let a, b, c represent real numbers; x, y, z elements of a fixed Banach space B; f, g, h continuous linear functionals on B, that is, elements of B^*; and X, Y, Z elements of B^{**}. Let us say that X is the reflection of x in B^{**}, and x the reflection of X in B, if $X(f) = f(x)$ for all f. Then B is called reflexive if the reflection of B in B^{**} is the whole of B^{**}.

Throughout this paper $\{x_n\}$ *will represent a basis for* B [1; 110] *and* $\{f_n\} \subset B^*$ *the sequence of functionals which, with* $\{x_n\}$, *forms a biorthogonal set.*

Lemma 1. *For each* $Y \in B^{**}$, $\sup_n \| \sum^n Y(f_k)x_k \| < \infty$. *Conversely, if* $\{a_n\}$ *is such that* $\sup_n \| \sum^n a_k x_k \| < \infty$, *there exists* $Y \in B^{**}$ *with* $Y(f_k) = a_k$.

The truth of this lemma reduces the proof of Theorem 10 in [2; 978] to its last two lines.

The direct part of the lemma yields to repeated applications of the Banach-Steinhaus theorems [1; 80] as follows. It is sufficient to show that $\sum^n Y(f_k)g(x_k)$ is bounded for each fixed Y, g. This, in norm, is not more than $\| Y \| \cdot \| \sum^n g(x_k)f_k \|$, thus it is sufficient to show that $\sum^n g(x_k)f_k$ is bounded. For this it is sufficient to show that $\sum^n g(x_k)f_k(z)$ is bounded for each fixed $z \in B$. But this series even converges (to $g(z)$).

The converse follows from Theorem 5 [1; 57] and the observation that for any scalars $t_1, t_2, \cdots, t_r$,

$$\left| \sum^r t_i a_i \right| = \left| \sum_{i=1}^r t_i f_i \left(\sum_{k=1}^r a_k x_k \right) \right|$$

$$\leq \left\| \sum^r t_i f_i \right\| \cdot \left\| \sum^r a_k x_k \right\| < M \cdot \left\| \sum^r t_i f_i \right\|.$$

Lemma 2. *For each* $h \in B^*$, $\sup_n \| \sum^n h(x_k)f_k \| < \infty$. *Conversely, if* $\{a_n\}$ *is such that* $\sup_n \| \sum^n a_k f_k \| < \infty$, *there exists* $h \in B^*$ *with* $h(x_k) = a_k$.

The proof follows the details of the proof of Lemma 1.

Received March 10, 1951; revision received September 17, 1951.