INTEGRABILITY OF TRIGONOMETRIC SERIES. I.

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1. Let $\sum a_n$ be an absolutely convergent series of real numbers and

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx, \tag{1.1}$$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx. \tag{1.2}$$

We are concerned with the existence of the Cauchy limits

$$\int_{\to 0} x^{-1} g(x)\, dx, \tag{1.3}$$

$$\int_{\to 0} x^{-1} f(x)\, dx. \tag{1.4}$$

We shall show that, in the first place, (1.3) always exists, but not necessarily as a Lebesgue integral (for a counterexample see Titchmarsh [3; 170-171]); of course (1.3) is a Lebesgue integral if $g(x) \geq 0$ in a neighborhood of 0. Second, since $f(x)$ is continuous an obvious necessary condition for the existence of (1.4) is

$$f(0) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n = 0.$$

If $f(0) = 0$, we shall show that a necessary and sufficient condition for the existence of (1.4) is the convergence of

$$\sum_{n=1}^{\infty} n^{-1}\left(\frac{1}{2} a_0 + \sum_{k=1}^{n} a_k\right) = - \sum_{n=1}^{\infty} n^{-1} \sum_{k=n+1}^{\infty} a_k . \tag{1.5}$$

If the a_k are ultimately of one sign, this is equivalent to the convergence of $\sum a_k \log k$; it can be shown that in this case (1.4) is a Lebesgue integral.

Theorems of this kind are sometimes useful for showing that a given function cannot have an absolutely convergent Fourier series. Thus for example an odd function of the form $h(x) = -1/\log x + xp(x)$ near $x = 0$, where $p(x)$ is an integrable function, cannot have an absolutely convergent Fourier series (as follows also from a result of Sz.-Nagy [2] if $h(x)$ is concave and increasing).

Our theorems are equivalent to other theorems which deal, not with absolutely convergent Fourier series, but with formal trigonometric series satisfying

$$\sum_{k=1}^{\infty} | \Delta a_k | < \infty, \qquad a_k \to 0. \tag{1.6}$$

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