CHARACTERISTIC VECTORS FOR A PRODUCT OF n REFLECTIONS

By J. S. Frame

The problem of this note is to determine in a specified coordinate system the characteristic vectors and characteristic equation of a matrix

$$(1) \qquad R = R_1 R_2 \cdots R_n ,$$

that is, an ordered product of n reflection matrices R_j, such that each R_j except R_1 is permutable with all its predecessors R_i ($i < j$), with the exception of just one predecessor $R_{j'}$, ($j' < j$) for which $R_{j'} R_j$ is of period 3.

We may represent the n reflections R_j by n nodes of a tree-like graph in which each node but the first is connected to exactly one predecessor, and we call connected nodes father (left) and son (right). The j-th node has exactly one "father" called the j'-th node, and has σ_j "sons", where σ_j may be 0, 1, 2, $\cdots$. We call the j-th node terminal, regular, or branching according as $\sigma_j = 0$, $= 1$, or > 1. Without changing the product R, its factors can always be arranged so that each regular node is directly followed by its only son and each branching node is directly followed by one of its sons. We assume this to be done, and divide up the product R into sections, such that each terminal node and each branching node ends a section. The k-th section consists of n_k nodes of which all but the last are regular, and we call two sections "father and son" if the last node of one section is linked to the first node of the other. The number s_k will denote the number of sons of the k-th section, and may be 0, 2, 3, $\cdots$. The symbols A_k and S_k will denote respectively the ancestors (father, grandfather, etc.) and the s_k sons (but not grandsons, etc.) of the k-th section, and will modify the summation signs that appear in the equations below.

The relationship of the R_j is described concisely by an upper triangular matrix T in which the j-th column has a 1 in the j-th row, a -1 in the j'-th row ($j' < j$), and 0's for all other entries. The symmetric matrix $\frac{1}{2}(T + T')$ is the matrix whose ij entry is the cosine of the angle between the i-th and j-th coordinate axes in an oblique coordinate system, in which the matrix R_j represents a reflection in the hyperplane through the origin perpendicular to the j-th axis.

Let λ be a characteristic root and let X be a characteristic row vector for the product matrix R. Coxeter [1] describes a factorization of R in the form $-T^{-1}T'$ and defines a second row vector Y such that

$$(2) \qquad X = -YT, \qquad \lambda X = YT',$$

obtaining in his characteristic equation $| \lambda T + T' | = 0$ a determinant whose order is equal to the number of nodes in the graph. It is our purpose to obtain

Received July 24, 1951.