THE PRODUCT OF THE GENERATORS OF A FINITE GROUP GENERATED BY REFLECTIONS

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In Euclidean n-space, every finite group generated by reflections leaves at least one point invariant, and thus may be regarded as operating on a sphere. It has for its fundamental region a spherical simplex whose dihedral angles are submultiples of π, say π/p_{ik} [6; 597, 619], [10; 190]. Accordingly we can use as generators the reflections $R_1, R_2, \cdots, R_n$ in the bounding hyperplanes of this simplex. The product $R_1R_2 \cdots R_n$ has already been found useful in various ways [6; 606–617], [7], [11]. The n generators may be taken in any order, since the products in different orders are all conjugate [6; 602]. Most of the applications of $R_1R_2 \cdots R_n$ were concerned with its period, h. In the present paper we consider its characteristic roots

$$\omega^{m_1}, \omega^{m_2}, \cdots, \omega^{m_n},$$

where $\omega = e^{2\pi i/h}$ and the exponents m_i are certain integers which may be taken to lie between 0 and h. They are computed by a trigonometrical formula involving the periods, p_{ik}, of the products of *pairs* of generators. (The product of two reflections is simply a rotation.)

The point of interest is that the same integers occur in a different connection. It turns out that the order of the group is

$$(m_1 + 1)(m_2 + 1) \cdots (m_n + 1),$$

and that these factors $m_i + 1$ are the degrees of n basic invariant forms [2; Chapter XVII]. Moreover, when every p_{ik} is 2, 3, 4 or 6, so that the group is *crystallographic*, there is a corresponding continuous group, and the Betti numbers of the group manifold are the coefficients in the *Poincaré polynomial*

$$(1 + t^{2m_1+1})(1 + t^{2m_2+1}) \cdots (1 + t^{2m_n+1}).$$

Having computed the m's several years earlier [10; 221, 226, 234], I recognized them in the Poincaré polynomials while listening to Chevalley's address at the International Congress in 1950. I am grateful to A. J. Coleman for drawing my attention to the relevant work of Racah [16], which helps to explain the "coincidence"; also, to J. S. Frame for many helpful suggestions (such as his idea of using the matrix T in §1, see [8; 6]), to J. A. Todd for his conjecture that the Jacobian of the basic invariants will always factorize into linear forms

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