# NON-EXISTENCE OF ODD PERFECT NUMBERS OF THE FORM $3^{2\beta}\cdot p^{\alpha}\cdot s_1^{2\beta_1}s_2^{2\beta_2}s_3^{2\beta_3}$

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Euler's result [2; 514], [3; 14–15] that any odd perfect number $n$ has the form $n = p^{\alpha}q_1^{2\gamma_1}q_2^{2\gamma_2} \cdots q_t^{2\gamma_t}$, where $p, q_1, \cdots, q_t$ are primes and $p \equiv 1 \equiv \alpha \pmod 4$, was extended by Sylvester [8], if $q_1 = 3$, in which case he proved that $t \geq 4$. In that paper Sylvester stated that $t = 4$ is impossible; this statement is proved in the present paper.

The following notations are used: $a \mid b$ and $a \nmid b$ mean $a$ divides $b$ and $a$ does not divide $b$, respectively; $a \to b \pmod m$ means $a$ belongs to $b \pmod m$.

**Auxiliary lemmas.** Lemmas 1 and 2 are due to Brauer [1].

Lemma 1. *Let $q$ be a positive prime. The Diophantine equation $q^2 + q + 1 = y^m$ has no solution for $m > 1$.*

Lemma 2. *Let $r$ and $s$ be different positive integers and $p$ be a prime. The system of simultaneous Diophantine equations $x^2 + x + 1 = 3p^r$, $y^2 + y + 1 = 3p^s$, has no solutions in positive integers $x$, $y$.*

The word different can be stricken from the above lemma since $x^2 + x + 1 = y^2 + y + 1$ implies $x + y = -1$ unless $x = y$.

We set $f_j(x) = x^{j-1} + \cdots + x + 1$ and refer to it as a cyclotomic sum. If $j$ is a prime $p$, then $f_p(x)$ is the $p$-th cyclotomic polynomial. It is well known that the prime divisors of $f_p(x)$ are $p$ and primes of form $pz + 1$, but $p^2$ is never a divisor.

Results concerning factors of $f_j(x)$ are contained in

Lemma 3. *If $m$, $q$ and $s$ are integers, $t$ a prime, then*
I. *$m \mid s$ implies $f_m(x) \mid f_s(x)$.*
II. *If $q \equiv 1 \pmod t$, then $f_s(q) \equiv 0 \pmod t$ if and only if $t \mid s$.*
III. *If $q \to k > 1 \pmod t$, then $f_s(q) \equiv 0 \pmod t$ if and only if $k \mid s$.*

*Proof.* The proofs of I and II are obvious from the form $f_j(x) = (x^j - 1)\cdot(x - 1)^{-1}$. In III, $q^k - 1 \equiv 0 \pmod t$, $f_k(q) \equiv 0 \pmod t$, so that $k \mid s$ implies $f_s(q) \equiv 0 \pmod t$ by I. For the converse let $s = ky + z$, $0 \leq z < k$. If $z > 1$, $f_s(q) = f_{ky}(q) + q^{ky} + q^{ky+1} + \cdots + q^{ky+z-1} \equiv 0 + 1 + q + \cdots + q^{z-1} \pmod t$. Hence, $f_s(q) \equiv 0 \pmod t$ implies $f_z(q) \equiv 0 \pmod t$ which is impossible with $z < k$. Accordingly, $z = 0$ so that $k \mid s$.

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