A NOTE ON BASIC SETS OF POLYNOMIALS

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The object of this paper is to rectify an oversight in a paper by R. P. Boas, Jr. [1]. For notation and relevant results see also J. M. Whittaker [2].

We consider a simple set $\{p_n(z)\}$ of polynomials, such that

$$p_n(z) = \sum_{k=0}^{n-1} p_{nk} z^k + z^n, \tag{1}$$

$$| p_{nk} | \leq \beta_{n-k} . \tag{2}$$

We write

$$h(R) = \sum_{k=1}^{\infty} \beta_k R^{-k} \tag{3}$$

Boas denotes by $H_2(R)$ the space of functions f, regular in $| z | < R$, with

$$\int_0^{2\pi} | f(re^{i\theta}) |^2 \, d\theta$$

bounded for $r < R$, and proves the following result [1; 720, Theorem 3].

(A) If $h(R) < 1$, then every $f \in H_2(R)$ can be represented uniquely in the form

$$f(z) = \sum_{n=0}^{\infty} c_n p_n(z), \tag{4}$$

where the expansion converges (a) in mean square on $| z | = R$ and (b) uniformly and absolutely in $| z | \leq r < R$.

He then goes on to say

(B) "Since the expansion (4) is unique, it must coincide with Whittaker's basic series. The statement in (b) implies that the set $\{p_n(z)\}$ is effective in $| z | < R$; the uniqueness is a consequence."

That the statement (B) is false is shown by the following example.

$$p_0(z) = 1; \qquad p_n(z) = z^n + \frac{1}{2n^2} \qquad (n > 0);$$

$$\beta_k = \frac{1}{2k^2}; \qquad h(1) = \sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{\pi^2}{12} < 1.$$

The base thus satisfies the hypotheses of (A). However,

$$\pi_{nn} = 1, \qquad \pi_{n0} = -\frac{1}{2n^2}, \qquad \pi_{nk} = 0, \qquad \text{otherwise.}$$

Received June 11, 1951.