AN ESTIMATE OF THE ERROR IN TAUBERIAN THEOREMS FOR POWER SERIES

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1. **Introduction.** Hardy and Littlewood [1] essentially proved the following Tauberian theorem for power series. Let

$$f(t) = \sum_{n=0}^{\infty} a_n e^{-nt}, \tag{1.1}$$

where the series converges for $t > 0$, and let

$$a_n > -n^{\alpha-1}L(n) \qquad (n = 1, 2, \cdots), \tag{1.2}$$

where $\alpha \geq 0$, and where $L(u)$ is a slowly oscillating function, that is, a positive continuous function defined for $u \geq 1$ and such that

$$L(au)/L(u) \to 1 \text{ as } u \to \infty$$

for every $a > 0$. Finally, let

$$f(t) = t^{-\alpha}L(t^{-1})\{A_1 + \rho(t)\} \qquad (0 < t < 1),$$

where $\rho(t) \to 0$ as $t \downarrow 0$. Then

$$s_n = a_0 + a_1 + \cdots + a_n = n^{\alpha}L(n)(A_1\{\Gamma(\alpha + 1)\}^{-1} + \rho^*(n)),$$

where $\rho^*(n) \to 0$ as $n \to \infty$. Mr. Erdös suggested to me that it would be worth while to investigate what one can say about the order of magnitude of $\rho^*(n)$ $(n \to \infty)$ if the order of magnitude of $\rho(t)$ $(t \downarrow 0)$ is known. The results in this paper imply for example that if $L(u) \equiv 1$ and $\rho(t) = O(t)$, then $\rho^*(n) = O\{(\log n)^{-1} \log \log n\}$; the latter estimate certainly can not be improved beyond $\rho^*(n) = O\{(\log n)^{-1}\}$, even if instead of (1.2) it is required that

$$a_n = O\{n^{\alpha-1}L(n)\}, \tag{1.3}$$

where in this case $L(u) \equiv 1$.

The more general results derived in this paper will be formulated for the case that the above constant $A_1 = 0$. Then the restriction that α be ≥ 0 is unnecessary. Let $f(t)$ be defined by (1.1), where the series is assumed to be convergent for $t > 0$, and where the coefficients a_n satisfy (1.2) for some real number α and some slowly oscillating function $L(u)$. Let

$$f(t) = O\{t^{-\alpha}L(t^{-1})\omega(t)\} \qquad (0 < t < 1), \tag{1.4}$$

where $\omega(t) \downarrow 0$ as $t \downarrow 0$, and

$$\omega(ut)/\omega(t) < A_2^u \qquad (u \geq 1,\ 0 < t \leq u^{-1}), \tag{1.5}$$

Received April 18, 1951.