THE ESSENTIAL MULTIPLICITY FUNCTION

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Introduction. Radó and Reichelderfer [3] have defined the essential multiplicity function for continuous plane transformations. We wish to consider the following inverse problem. *When is a function the essential multiplicity function of some continuous plane transformation?* As this question appears to be rather difficult, the one-dimensional case is discussed first. The complete answer is given in 3.2. In 2.1, we consider a generalization of the essential multiplicity function to general (one-dimensional) transformations. Here also, a complete answer is given to the corresponding inverse question. Finally in 4.4 a partial result is given in the two-dimensional case.

1. Definitions and preliminary theorems.

1.1. Let E be a non-empty subset of $[0, 1]$. Let $f(x)$, $f^*(x)$, $x \in E$, be real-valued functions. Let E^* be a non-empty subset of E. Let $\rho(f, f^*, E^*) = \sup | f(x) - f^*(x) |$ for $x \in E^*$.

1.2. Let f, E, and E^* be as in 1.1, and let y_0 be finite. Then the *crude multiplicity* $N(y_0, f, E^*)$ is the number (possibly $+\infty$) of points in the set $f^{-1}(y_0) \cap E^*$.

This concept was introduced by Banach in [1].

1.3. Let f, e, and E^* be as in 1.1. Let $\phi(y_0, f, E^*)$ equal the supremum (possibly $+\infty$) of the positive integers k such that there exist $x_0 \in E^*, \cdots, x_k \in E^*$, for which (i) $x_0 < x_1 < \cdots < x_k$, and (ii) either $f(x_0) < y_0$, $f(x_1) > y_0$, $f(x_2) < y_0, \cdots$, or $f(x_0) > y_0$, $f(x_1) < y_0$, $f(x_2) > y_0, \cdots$. If there does not exist such a positive integer k, then let $\phi(y_0, f, E^*) = 0$.

1.4. Let E be as in 1.1, and let $y = f(x)$, $0 \leq x \leq 1$, be a continuous function. The *essential multiplicity* $\kappa(y_0, f, E)$ is the supremum (possibly $+\infty$) of the non-negative integers k such that there exists an $\epsilon(k, y_0, f) > 0$ for which it is true that if $y = f^*(x)$, $0 \leq x \leq 1$, is a continuous function such that $\rho(f, f^*, [0, 1]) < \epsilon(k, y_0, f)$, then $N(y_0, f^*, E) \geq k$.

1.5. If $E = [0, 1]$ the symbols $\rho(f, f^*)$, $N(y, f)$, $\phi(y, f)$ and $\kappa(y, f)$ will be used to denote $\rho(f, f^*, E)$, $N(y, f, E)$, $\phi(y, f, E)$, and $\kappa(y, f, E)$ respectively.

1.6. THEOREM. *If $y = f(x)$, $0 \leq x \leq 1$, is a real-valued continuous function, then $\phi(y, f) \equiv \kappa(y, f)$.*

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