LOCAL THEORY OF RESIDUES

By George Whaples

In 1934, Hasse [3] defined differentials and residues in fields of algebraic functions with an arbitrary perfect field (possibly of prime characteristic) as field of constants, and proved that the sum over all prime spots of the residues of any differential is zero. For one purely local lemma, his proof involves a quite difficult passage to characteristic zero and back; we present here a simple purely local proof in which this difficulty is avoided.

In the meantime, elegant global theories of residues have been created by Artin [unpublished], Chevalley [2], and Weil [6]. From each of these theories could be extracted a proof of the lemma; but it would be quite different from the proof given here.

1. **Local residues.** Let k be a field which is complete with respect to a non-Archimedean discrete valuation. Assume that the residue class field $\mathfrak{o}/(\pi)$ is perfect, i.e., has no inseparable extensions, where $\mathfrak{o}$ is the ring of k-elements of value at most 1, and π is a prime element. Assume further that k contains at least one subfield such that $\mathfrak{o}/(\pi)$ consists of the residue classes of the elements of this subfield, and that a certain one of these subfields, k_0, which we shall call the *local field of constants* of k, has been chosen once for all. (Under our assumption that the field of residue classes is perfect, a necessary and sufficient condition for the existence of such a subfield k_0 is that k and the field of residue classes have the same characteristic, and if this characteristic is a prime, then k_0 is unique; see [7; Theorems 7 and 8], and [5; Chapter 7]. But in the case of characteristic zero, one can easily construct examples where k_0 is not unique by taking a field of formal power series over a field which is itself transcendental over the rational field.) When k has prime characteristic and $\mathfrak{o}/(\pi)$ is a Galois field, then k_0 is just the set of all roots of unity in k, together with 0. Of course, $k_0 \cong \mathfrak{o}/(\pi)$ so k_0 is perfect. If K is algebraic of finite degree over k, we shall assume that the local field of constants K_0 of K is the algebraic closure of k_0 in K; it is easy to see from Hensel's lemma and the fact that k_0 is perfect that the elements of this algebraic closure generate $\mathfrak{O}_K/(\Pi_K)$.

If τ is any element of k with $|\tau| < 1$, then every Laurent series

$$\sum_{\mu=-m}^{\infty} a_\mu \tau^\mu \qquad (a_\mu \in k_0) \tag{1}$$

converges in k; let $k_0\langle\tau\rangle$ denote the subfield consisting of all elements (1). If π is any prime element of k then $k = k_0\langle\pi\rangle$.

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