# A CLASS OF SPIRALS

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1. **Introduction.** In certain recent work in gas dynamics [3], it has become of importance to investigate the geometric behavior of curves whose equations in the complex plane are expressed in the form

$$z = \int_0^t [a_1(u) + ia_2(u)]e^{ib(u)}\, du \qquad (0 \le t < a \le \infty),$$

where $a_1$ , $a_2$ , and $b$ have certain known properties. In this paper such an investigation is made for the case when the curvature of the corresponding curve is monotone. The case $a_1 \ge 0$, $a_2 \equiv 0$, $b(t) \equiv t$, and $a = \infty$ is first considered (§§2, 3, 4) with particular emphasis on the behavior of the curve as $t$ becomes infinite. It then appears (§§5, 6) that the general case can be reduced to this special case or to a generalization which can be treated by essentially similar methods.

2. **Two types of spirals.** Consider the curve $C$ whose equation in the complex plane is given by

$$z = \int_0^t r(u)e^{iu}\, du \qquad (0 \le t < \infty), \tag{2.1}$$

where $r(t)$ satisfies the following conditions:

(2.2)
(a) $r(t)$ is continuous, non-negative, and non-decreasing on $(0, \infty)$ and vanishes at most for the single value $t = 0$.
(b) $\lim_{t \to \infty} r(t) = \infty$
(c) $r(t)$ is not constant on any interval $(t_0 , t_0 + 2\pi)$.

It is verified at once that $t$ is the slope angle for the curve, while $r(t)$ is the radius of curvature. The curve thus has monotone decreasing curvature. The circles of curvature therefore form an expanding family, each one containing or being identical with each preceding one [2; Lemma 2.5]. Moreover, such a curve is simple unless it contains a complete circle [2; Corollary 2.5.2] which is here impossible by condition (2.2c). Hence, $C$ is a simple outwinding spiral. The origin $O$ is on the circle of curvature for $t = 0$ and interior to all other circles, so the radius vector $OP$ to an arbitrary point $P$ on the curve turns always in a counterclockwise sense. If $\alpha$ is the slope angle of $OP$, it is easily seen that $\alpha < t < \alpha + \pi$ so $\alpha$ and $t$ become infinite together, and the curve winds infinitely often about 0.

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