CONTRACTIONS IN A HYPERBOLIC SPACE

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In this paper we prove two known function theoretic results, namely, Julia's theorem and Wolff's theorem, in an abstract metric setting. Both these theorems concern fixed points of a holomorphic function f mapping an open circle of the full complex plane into itself. If we consider such an open circle as a conformal Klein model of hyperbolic plane geometry, then, according to Schwarz's lemma or Pick's theorem, the mapping f is either a motion or a contraction. We abstract certain purely metric properties of hyperbolic plane geometry with compactifying ideal points adjoined and call any space possessing these properties a hyperbolic space. In such a space Wolff's theorem may be formulated as follows: Let f be a contraction of a hyperbolic space into itself; there then exists a point, which may be either an ordinary point in the space or an ideal point adjoined to it, to which every orbit of f converges. Julia's theorem in abstract form is used in proving this result.

Hyperbolic space. A metric space which becomes sequentially compact after certain ideal limit points have been adjoined to it will be called a hyperbolic space if its distance function satisfies the following conditions with regard to approach by ordinary points in the space to the adjoined ideal limit points: an ideal point is infinitely far from an ordinary point; two distinct ideal points are infinitely far apart; distances from two ordinary points to an ideal point can be unambiguously compared; an ideal point is in comparison infinitely farther from another ideal point than from an ordinary point.

This introductory description of a hyperbolic space is amplified in more detail below.

Let H and $H^{\cdot}$ be disjoint abstract sets with union H^*. The elements of H^* will be called points; when we want to be more specific we call a point in H ordinary and a point in $H^{\cdot}$ ideal. A given point $x^* \in H^*$ is then either ordinary or ideal: if $x^* \in H$ we write $x^* = x$, and if $x^* \in H^{\cdot}$ we write $x^* = x^{\cdot}$.

Let $\mathfrak{L}$ be a set of pairs $(\{x_\nu\}, x^*)$, where $x^* \in H^*$ and $\{x_\nu\}$ represents a sequence of ordinary points $x_\nu \in H$ as ν runs in order over the set N of all natural numbers $0, 1, 2, \cdots$. We shall write $x_\nu \to x^*$ by way of abbreviating the statement $(\{x_\nu\}, x^*) \in \mathfrak{L}$ and say that the sequence $\{x_\nu\}$ converges to x^* and that x^* is a limit point of $\{x_\nu\}$. An infinite subset N' of N, indicated hereafter by the notation $N' \prec N$, selects from any given sequence $\{x_\nu\}$ a certain subsequence which we denote by $\{x_\nu \text{ via } N'\}$; for convenience we write $x_\nu \to x^*$ via N' instead of x_ν via $N' \to x^*$. Note that $\{x_\nu \text{ via } N\}$ is the same sequence as $\{x_\nu\}$ and that $x_\nu \to x^*$ via N means the same as $x_\nu \to x^*$. The set of all subsequential limit points of a

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