SOME EXTENSIONS OF PIRANIAN'S THEOREM

By R. Wilson

1. **Introduction.** Let the Taylor expansion of the function $f(z)$, regular in the neighborhood of the origin, take the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots . \tag{1}$$

In the case in which $f(z)$ is meromorphic inside a given circle Piranian [8] has obtained the results set out below.

Theorem 1. *If $f(z)$ of (1) is regular in the circle $|z| = |z_0|$ except for poles at the points $z_\nu(\neq 0)$, $\nu = 1, 2, \cdots, r$, of the respective multiplicities m_ν, and if the only singularity of $f(z)$ on the circle $|z| = |z_0|$ is an ordinary algebraic-logarithmic singularity of weight $[\sigma, k]$ at the point z_0, then*

$$\begin{aligned}\sigma &= 1 + \lim_{n\to\infty} \frac{\log |D_{n,p}| + n \log (|z_0| \prod_1^r |z_\nu|^{m_\nu})}{\log n} \\ k &= \lim_{n\to\infty} \frac{\log |D_{n,p}| + n \log (|z_0| \prod_1^r |z_\nu|^{m_\nu}) - (\sigma - 1) \log n}{\log (\log n)},\end{aligned} \tag{2}$$

where $\sum_1^r m_\nu = p$ and

$$D_{n,p} = |a_{n+\mu+\nu}| \qquad (\mu, \nu = 0, 1, \cdots, p). \tag{3}$$

Generalizations of this result are obtained for the cases in which the singularity at $z = z_0$ is a transcendental algebraic-logarithmic singularity [3] (which includes the ordinary as well as the generalized algebraic-logarithmic singularity of Jungen [5; §7] as special cases) and in which it is an isolated essential point of finite exponential order [6; §3], [9; 203]. The case in which there are several algebraic-logarithmic singularities on $|z| = |z_0|$ is also considered.

A general method is developed in §3 which depends on the lemma proved in §2. The applications referred to above are seen in §4 to arise as special cases of the general process.

2. **The basic lemma.** The proofs of Piranian's and the associated theorems depend on the following lemma.

Lemma 1. *If $f(z)$ of (1) is regular in the circle $|z| = |z_0|$ except for poles at the points $z_\nu(\neq 0)$, $\nu = 1, 2, \cdots, r$, of the respective multiplicities m_ν, with $\sum_1^r m_\nu = p$,*

Received October 7, 1949; in revised form, May 16, 1951.