TOPOLOGIES DEFINED BY BOUNDED SETS

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1. **Introduction.** It is well known that in some linear spaces certain topological concepts can be characterized with the help of a notion of boundedness. For instance, a functional (additive, numerical valued function) on a Banach space is continuous if and only if it is bounded on the unit sphere of the Banach space. Mackey [10,11] and the author [3] have shown that similar situations can arise in much more general linear spaces. Hu [4] has discussed boundedness as a fundamental concept in a non-linear space. In the present paper, we suppose given a vector space S and a family $\mathcal{B}$ of subsets of S satisfying certain axioms and called the bounded subsets of S. We inquire into the existence and properties of a topology in S in which the sets of $\mathcal{B}$ are bounded in a topological sense.

In §2, we give the axioms concerning $\mathcal{B}$ and explain the notation which we will use.

§3 develops some algebraic consequences of our axioms. Theorem 1 shows that an n-dimensional set is in $\mathcal{B}$ if and only if it can be enclosed in an n-dimensional cube.

A topology in S is defined in §4 which makes all the sets of $\mathcal{B}$ topologically bounded. The space S may fail to be a Hausdorff space and the addition in S may not be continuous in both arguments jointly. However, the instances in which S is a normed linear space are easily characterized (Theorem 3).

The collection $\mathfrak{I}$ of all subsets of S which are topologically bounded is discussed in §5. It is proved (Lemma 6) that $\mathfrak{I} \supset \mathcal{B}$ and (Theorem 5) that $\mathfrak{I}$ and $\mathcal{B}$ define the same topology in S. The sets of $\mathfrak{I}$ are characterized in two ways (Lemma 7, Theorem 4).

Convergence properties of S are given in §6 and several examples and counter examples appear in §7.

2. **Notation, axioms and definitions.** The definitions for some of the standard terms are given at the end of this section.

Let $S = \{\theta, a, b, x, y, \cdots\}$ be a vector space over the real number field with zero vector θ. We use Roman capitals for subsets of S, script $\mathcal{B}$ and $\mathfrak{I}$ for collections of subsets of S. The letters i, j, k, n index countable sets; α, β, γ index arbitrary (sometimes directed) systems; other Greek letters (except θ) denote real scalars. The symbols $+$ and $\sum$ indicate linear additions; $\bigcup$ is used for point set union. A prime is used for point set complementation and for negation with the symbol ε.

Let $\mathcal{B}$ be any collection of subsets of S satisfying the following axioms.

(B1). For any $x \in S$, $\{x\} \in \mathcal{B}$.

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