SEPARATION IN NON-SEPARABLE SPACES

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Axioms 0 and 1 of R. L. Moore's *Foundations of Point Set Theory* [1] are as follows.

Axiom 0. *Every region is a point set.*

Axiom 1. *There exists a sequence* $G_1, G_2, G_3, \cdots$ *such that* (1) *for each* n, G_n *is a collection of regions covering* S, (2) *for each* n, G_{n+1} *is a subcollection of* G_n, (3) *if* R *is any region whatsoever,* X *is a point of* R *and* Y *is a point of* R *either identical with* X *or not, then there exists a natural number* m *such that if* g *is any region belonging to the collection* G_m *and containing* X *then* $\bar{g}$ *is a subset of* $(R - Y) + X$, (4) *if* $M_1, M_2, M_3, \cdots$ *is a sequence of closed point sets such that, for each* n, M_n *contains* M_{n+1} *and, for each* n, *there exists a region* g_n *of the collection* G_n *such that* M_n *is a subset of* $\bar{g}_n$, *then there is at least one point common to all the point sets of the sequence* $M_1, M_2, M_3, \cdots$.

If Axiom 1_3 denotes Axiom 1 above with part (4) deleted, there does not exist a space, satisfying Axiom 1_3, which is a subspace of any space, satisfying all of Axiom 1, which is not separable and does not contain uncountably many mutually exclusive domains [2]. However, there does exist a space, satisfying Axioms 0 and 1_3, which is not separable and does not contain uncountably many mutually exclusive domains [3]. This example [3] strongly uses the fact that no region has a boundary point. It is easy to connect the points of this space without destroying the properties described, but the questions arise as to whether such a space could be locally connected and by what types of point sets could each two points of such a space be separated. This paper will answer some of these questions.

Theorem 1. *There is a locally connected space, satisfying Axioms* 0 *and* 1_3, *which is not separable and does not contain uncountably many mutually exclusive domains.*

Proof. For each positive integer x, let I_x denote the sequence of points whose rectangular coordinates are $(x,1)$, $(x,\frac{1}{2})$, $(x,1/3)$, $\cdots$. There is an uncountable well-ordered sequence β of rays such that (1) if Y is a ray of β there is a sequence $Y_1, Y_2, Y_3, \cdots$, where, for each i, Y_i belongs to I_i, and Y is made up of the points of the straight line intervals $Y_1Y_2, Y_2Y_3, Y_3Y_4, \cdots$, (2) if Z follows Y in β there is a positive integer n such that, for i greater than n, $Z \cdot I_i$ is below $Y \cdot I_i$, and (3) no term of β is preceded by uncountably many others.

An essential notion in the definition of point for the desired space will be the idea of a *breakdown*. A sensed pair (Y,e) is a *breakdown* of *order* i of the finite set

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