PROPERTIES OF LINEAR TRANSFORMATIONS PRESERVED UNDER ADDITION OF A COMPLETELY CONTINUOUS TRANSFORMATION

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1. **Introduction.** In this paper we consider various properties of linear continuous transformations on a Banach space which are possessed by transformations of the form $I + U$, where I is the identity and U is completely continuous. We investigate several of the principal properties given by Riesz [7] and Schauder [8] and certain related properties obtaining characterizations of the classes of homomorphisms (see §2 for notation) satisfying these properties. The basic classes are the homomorphisms with finite-dimensional null-space and those with finite deficiency. These were investigated by Dieudonné [4]. Here in §3 we characterize these and also show that these properties are preserved upon addition of a completely continuous linear transformation. In §4 the relation of this theory to an open question of Banach is pursued.

In §5 we discuss certain other properties involving projections. Upon specialization, this discussion gives necessary and sufficient conditions on a bounded linear transformation T in order that there exists another transformation V such that TV (or VT) is of the form $I + U$.

It is felt that the present work gives a more complete picture of the Riesz theory [7] and extends its scope. The following example also illustrates this. One of the principal properties for $I + U$ is that of [7; Satz 2] which states that there exists an integer m such that for all $n \geq m$ the null-spaces of $(I + U)^n$ are identical. This property fails if I is replaced by an isomorphism onto. For let $\mathfrak{X}$ be the Banach space l_1, say. For $x = (\xi_1, \xi_2, \cdots)$, $y = (\eta_1, \eta_2, \cdots)$, define $T(x) = y$ by the rule $\eta_1 = \xi_2$, $\eta_{2n} = \xi_{2n+2}$, $n = 1, 2, \cdots$ and $\eta_{2n+1} = \xi_{2n-1}$, $n = 1, 2, \cdots$, i.e.,

$$T(x) = (\xi_2, \xi_4, \xi_1, \xi_6, \xi_3, \xi_8, \cdots).$$

It is clear that T is an isomorphism of $\mathfrak{X}$ onto $\mathfrak{X}$ (indeed an equivalence). In the above notation we define the transformation U by the rule $U(x) = y$, where $\eta_1 = \xi_2$, $\eta_i = 0$, $i \neq 1$. U is completely continuous. Consider $V = T - U$. Let $x_n = \{\xi_i^n\}$ be defined by the rule $\xi_{2n}^n = 1$, $\xi_i^n = 0$, $i \neq 2n$. We show, by induction, that, for each n, $V^n(x_n) = 0$ but $V^{n-1}(x_n) \neq 0$. The statement clearly holds for $n = 1$. Observe that for $m > 1$, $V(x_m) = x_{m-1}$. Then if the result holds for $m \geq 1$, for $m + 1$ we have

$$V^{m+1}(x_{m+1}) = V^m V(x_{m+1}) = V^m(x_m) = 0,$$

$$V^m(x_{m+1}) = V^{m-1} V(x_{m+1}) = V^{m-1}(x_m) \neq 0.$$

This shows that the cited theorem of Riesz fails for transformations of the type $T + U$, where T is an isomorphism onto and U is completely continuous.

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