THE ZEROS OF APPROXIMATING POLYNOMIALS AND THE CANONICAL REPRESENTATION OF AN ENTIRE FUNCTION

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1. **Introduction.** By C. Weierstrass' factorization theorem every entire function $f(z) \not\equiv 0$ may be written in canonical form as follows:

$$f(z) = \exp\{s(z)\}z^m \prod_p \left(1 - \frac{z}{z_p}\right) \exp\left\{\frac{z}{z_p} + \cdots + \frac{z^{h_p}}{h_p z_p^{h_p}}\right\}, \tag{1.1}$$

where $s(z)$ is an entire function and the h_p are integers ≥ 0 such that

$$\sum_p \left| \frac{z}{z_p} \right|^{h_p+1} \tag{1.2}$$

is convergent for all z. J. Hadamard showed that for an entire function of finite order $\leq \sigma$ there exists an integer $h \leq \sigma$ such that (1.2) is convergent for all z if we choose all h_p equal to h; by choosing all h_p equal to such an h, $s(z)$ reduces to a polynomial of degree $\leq \sigma$. (See [9; Chapter VIII].)

It was discovered by E. Laguerre that there is a certain connection between the distribution of the zeros of polynomials approximating to an entire function and its properties as exhibited by its canonical form(s). Laguerre's results and subsequent investigations by G. Pólya, E. Lindwart, N. Obrechkoff and others led me to the following problem.

Let R be a set in the z-plane. Let $C(R)$ denote the class of R-functions, that is, the class of those entire functions $\not\equiv 0$ that may be obtained as the limit of a sequence of polynomials $f_n(z)$, all of whose zeros lie in R, and which sequence converges uniformly in every bounded domain. Is it possible to characterize the class $C(R)$, using the properties of R?

Using the above terminology we may say that Laguerre investigated the class $C(R)$ in the case that R is a half-line or a line; Pólya and Obrechkoff investigated $C(R)$ in the case that R is a sector of aperture $<\pi$, and equal to π, respectively. A survey of these results may be found in Obrechkoff's monograph [7].

$C(R)$ is obviously a multiplicative class. It is a closed class, if we define convergence as uniform convergence in every bounded domain. Hence for any set R, $C(R) \equiv C(\bar{R})$, where $\bar{R}$ is the closure of R. Henceforth we shall only consider closed sets R.

If R is bounded $C(R)$ is simply the class of the R-polynomials, that is, the class of all polynomials whose zeros lie in R (see §3). For unbounded sets R the asymptotic directions φ of R are of importance. They are the angles φ for which R contains a sequence $\{z_n\}$ such that $|z_n| \to \infty$, $\arg z_n \to \varphi$. By

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