GEODESIC COORDINATES AND REST SYSTEMS FOR GENERAL LINEAR CONNECTIONS

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Let $x_1, x_2, \cdots, x_n$ denote the "original variables", i.e., the coordinates of a point in an n-dimensional manifold L_n with an (asymmetric) linear connection. Then the components $\Gamma^{\lambda}_{\alpha\beta}(x_1, \cdots, x_n)$ of this connection transform according to [3; 327]

$$\overline{\Gamma}^{l}_{ik} = \Gamma^{\lambda}_{\alpha\beta}\frac{\partial x_\alpha}{\partial \bar{x}_i}\frac{\partial x_\beta}{\partial \bar{x}_k}\frac{\partial \bar{x}_l}{\partial x_\lambda} + \frac{\partial^2 x_\lambda}{\partial \bar{x}_i \partial \bar{x}_k}\frac{\partial \bar{x}_l}{\partial x_\lambda}. \tag{1}$$

From (1) we get the transformation formula for the torsion tensor $S^{\lambda}_{\alpha\beta}$ of the L_n :

$$\overline{S}^{l}_{ik} = S^{\lambda}_{\alpha\beta}\frac{\partial x_\alpha}{\partial \bar{x}_i}\frac{\partial x_\beta}{\partial \bar{x}_k}\frac{\partial \bar{x}_l}{\partial x_\lambda}, \qquad S^{\lambda}_{\alpha\beta} = -S^{\lambda}_{\beta\alpha} = \frac{1}{2}(\Gamma^{\lambda}_{\alpha\beta} - \Gamma^{\lambda}_{\beta\alpha}) = \Gamma^{\lambda}_{[\alpha\beta]} . \tag{2}$$

It is well known [2; 105] that L_n permits the introduction of geodesic coordinate systems if and only if the torsion tensor vanishes either locally or everywhere. Indeed from $\Gamma^{\lambda}_{\alpha\beta} = 0$ (in geodesic coordinates), the symmetry of the second derivatives in the suffixes i and k, and (1), it follows that the connection $\overline{\Gamma}^{l}_{ik}$ is symmetric; thus from (2), $\overline{S}^{l}_{ik} = 0$ in this (and so in every) system.

After an affine parameter s has been chosen the geodesics of L_n satisfy the differential equations

$$(x^i)'' + \Gamma^{i}_{\alpha\beta}(x^\alpha)'(x^\beta)' = 0. \tag{3}$$

Denote the symmetric part of the connection $\Gamma^{i}_{\alpha\beta}$ by $\Gamma^{i}_{(\alpha\beta)}$, and take into account the symmetry of the products $(x^\alpha)'(x^\beta)'$. Then we get from (3) and (2)

$$(x^i)'' + \Gamma^{i}_{\alpha\beta}(x^\alpha)'(x^\beta)' = (x^i)'' + (\Gamma^{i}_{(\alpha\beta)} + S^{i}_{\alpha\beta})(x^\alpha)'(x^\beta)'$$

$$= (x^i)'' + \Gamma^{i}_{(\alpha\beta)}(x^\alpha)'(x^\beta)' = 0, \tag{4}$$

$$\Gamma^{i}_{(\alpha\beta)} = \tfrac{1}{2}(\Gamma^{i}_{\alpha\beta} + \Gamma^{i}_{\beta\alpha}),$$

because $S^{i}_{\alpha\beta}(x^\alpha)'(x^\beta)'$ vanishes as a consequence of the skew-symmetry of the torsion tensor. This means that *the geodesics of an L_n are independent of the torsion* [2; 97]. If we introduce geodesic coordinates with respect to $\Gamma^{i}_{(\alpha\beta)}$ [2; 100], then from (4), we get, locally in these coordinates, $\Gamma^{i}_{(\alpha\beta)} = 0$, and thus $(x^i)'' = 0$. Following common usage in theory of relativity we call such systems *rest systems.*

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