REGIONS OF FLATNESS FOR ANALYTIC FUNCTIONS AND THEIR DERIVATIVES

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1. **Introduction.** The notion of "flatness" of an analytic function was first considered by J. M. Whittaker [5], [6], [7] and later studied by MacIntyre [1], [2], Valiron [4] and others. In the works of these writers the regions of a set are called "flat" regions of an analytic function if in these regions the maximum and the minimum of the modulus of the function are in some sense of the same order. It is evident that such properties of flatness are necessarily properties relative to a set of regions and hence, by mapping the regions of the set on a fundamental region, depend upon the properties of a family of functions in a region.

In the light of these considerations there is given in this paper a precise definition of regions of flatness in the case in which the set of regions is a set of circles. The notion of flatness as here defined is somewhat the same in character as that of Whittaker. With the definition of flatness here introduced, there is then stated a necessary and sufficient condition that the circles of a set shall be regions of flatness for a function $f(z)$ which is holomorphic in a domain containing the circles. As the principal result of this paper there are given conditions under which the circles of a set shall be regions of flatness for the derivatives of a function when it is known that they are regions of flatness for the function. These results are obtained by means of the theory of normal families and, in the case of the higher derivatives, by the theory of kernels as developed by Mandelbrojt [3].

2. **Definition of regions of flatness and a fundamental theorem.** Let $C(\alpha, R)$ denote the circle $| z - \alpha | < R$. Consider the set S of circles $C(\alpha, R)$, where α is complex, R real and positive, and the pair (α, R) varies over a set Ω of values. Suppose a function $f(z)$ is holomorphic in a domain which contains each circle of the set S. For a given real number η belonging to the interval $(0, 1)$, let

$$M(\alpha, R; \eta) = \operatorname*{Max}_{z \varepsilon C^*(\alpha, \eta R)} | f(z) |$$

$$m(\alpha, R; \eta) = \operatorname*{Min}_{z \varepsilon C^*(\alpha, \eta R)} | f(z) |$$

$$L(\alpha, R; \eta) = \operatorname{Min} \left\{ \left| \frac{\log M(\alpha, R; \eta)}{\log m(\alpha, R; \eta)} \right|, \frac{M(\alpha, R; \eta)}{m(\alpha, R; \eta)} \right\},$$

where $C^*(\alpha, R)$ denotes the closed circle $| z - \alpha | \leq R$.

Received July 26, 1949.