THE PARTIAL SUMS OF SECOND ORDER OF THE GEOMETRIC SERIES

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Introduction

1. The positive (non-negative) character of trigonometric polynomials with real coefficients, as the variable ranges over *all* real values, has been discussed by various authors. It is somewhat less obvious to study the positivity of trigonometric polynomials as the variable ranges over a fixed interval which is a part of the period. We mention two instances where questions of this kind enter.

(a) Let $f(\varphi)$ be a Riemann-integrable function with period 2π, and let α be a fixed real value. We consider the integral

$$T_n = T_n(\alpha) = \int_{-\pi}^{\pi} f(\varphi)p_n(\varphi)\, d\varphi, \tag{1}$$

where the "kernel" $p_n(\varphi) = p_n(\varphi, \alpha)$ is a given periodic function of period 2π which is continuous and depends on the parameter n. We are interested in conditions concerning $p_n(\varphi)$ involving the relation

$$\lim_{n\to\infty} T_n = f(\alpha) \tag{2}$$

for all functions $f(\varphi)$ which are continuous at the point $\varphi = \alpha$. The classical conditions of Lebesgue [4; 63] are as follows:

(I) $p_n(\varphi) \geq 0$ for all real φ,

(II) $\int_{-\pi}^{\pi} p_n(\varphi)\, d\varphi = 1$,

(III) $\lim_{n\to\infty} p_n(\varphi) = 0$ uniformly in every closed interval not containing any point $\varphi \equiv \alpha \pmod{2\pi}$.

It is almost obvious that condition (I) can be replaced by the following:

(I′) $p_n(\varphi) \geq 0$ for sufficiently large n and for all φ from a certain fixed interval containing $\varphi = \alpha$ in its interior [3].

Frequently $p_n(\varphi)$ is a trigonometric polynomial; thus we are led to the discussion of the positivity of such a polynomial in a fixed interval.

(b) Various partial sums of higher order of the geometric series

$$1 + z + z^2 + \cdots + z^n + \cdots \tag{3}$$

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