IDENTITIES INVOLVING THE COEFFICIENTS OF CERTAIN DIRICHLET SERIES

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1. Researches on the order of magnitude of various numerical functions have led to the discovery of many interesting identities in the analytic theory of numbers [1; 815–826], [5], [10]. Although they originate from a large variety of problems, these identities have certain features in common; namely, they contain infinite series of Bessel functions and the numerical functions involved are the coefficients of Dirichlet series which have functional equations. In this paper we present simplified proofs of many of these identities and the method used allows us to derive some new ones, most of our results occurring as special cases of the following general theorem concerning certain Dirichlet series.

Theorem 1. *For* $\lambda > 0$, $\kappa > 0$, $\gamma = \pm 1$, *let* $\phi(s) = \phi(\sigma + it)$ *satisfy the following conditions (in Hecke's* [4] *terminology,* $\phi(s)$ *has signature* $(\lambda, \kappa, \gamma)$*):*

(i) $\phi(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, *which converges absolutely for* $\sigma > \kappa$,

(ii) $\left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s)\phi(s) = \gamma\left(\frac{2\pi}{\lambda}\right)^{s-\kappa}\Gamma(\kappa - s)\phi(\kappa - s)$,

(iii) $(s - \kappa)\phi(s)$ *is an integral function of finite genus.*

Then, denoting the residue of $\phi(s)$ *at* $s = \kappa$ *by* ρ, *we have the identity:*

$$\frac{1}{q!}\sum_{n=0}^{x} a(n)(x-n)^q = \rho\frac{\Gamma(\kappa)}{\Gamma(\kappa+q+1)}x^{\kappa+q} + \gamma\left(\frac{\lambda}{2\pi}\right)^q x^{\frac{1}{2}(\kappa+q)}\sum_{n=1}^{\infty}\frac{a(n)}{n^{\frac{1}{2}(\kappa+q)}}J_{\kappa+q}\left(\frac{4\pi}{\lambda}(nx)^{\frac{1}{2}}\right), \tag{1.1}$$

where $x > 0$, q *is a positive integer,* $J_\nu(z)$ *is the ordinary Bessel function of order* ν, *and* $a(0)$ *is defined to be* $-\phi(0)$. *The series of Bessel functions is absolutely convergent if* $q > \kappa - \frac{1}{2}$.

Proof. We begin with the well-known integral

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{y^s}{s(s+1)\cdots(s+q)}\,ds = \begin{cases} 0 & \text{if } y \le 1 \\ \frac{1}{q!}\left(1 - \frac{1}{y}\right)^q & \text{if } y \ge 1 \end{cases} \qquad (c > 0,\ y > 0).$$

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