ASYMPTOTIC RELATIONS IN TOPOLOGICAL GROUPS

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The following theorem has been proved by Kawada [2].

Let G be an additive Abelian connected locally compact group, let ν be Haar measure in G, let E be a non-vacuous open subset of G such that cls E (*where* cls E *denotes the closure of* E) *is compact and let $x \in G$. Then*

$$\lim_{n\to\infty} \frac{\nu((nE) \cap (nE + x))}{\nu(nE)} = 1. \tag{1}$$

In connection with the study of generalized dynamical systems (see Bernard [1]) it would be desirable to have (1) available under less restrictive hypotheses. The purpose of this paper is to show that (1) remains valid if it is no longer assumed that G is connected, but it is assumed instead that some translate of E generates G.

The additive group of integers (reals) with its discrete (natural) topology is denoted by $\mathcal{I}(\mathcal{R})$.

Let G be an Abelian group which is generated by some compact neighborhood of the identity. A known structure theorem (see Weil [3; 110]) states that G is isomorphic to a Cartesian product $\mathcal{I}^m \times \mathcal{R}^p \times C$ for some non-negative integers m, p and some compact Abelian group C. Since the presence of C causes no difficulty in the derivation of our result, for the present we shall be concerned only with the group $\mathcal{I}^m \times \mathcal{R}^p$.

Parentheses will be used only as symbols of grouping and not merely to enclose the argument of a function. Where the grouping is obvious, parentheses may be omitted.

Let m, p be non-negative integers and let $q = m + p$. We note that $\mathcal{I}^q \subset \mathcal{I}^m \times \mathcal{R}^p \subset \mathcal{R}^q$. Haar-Lebesgue measure in $\mathcal{I}^m(\mathcal{R}^p)(\mathcal{R}^q)$, denoted by $\sigma_0(\sigma_1)(\sigma)$, is normalized so that the measure of a point (unit cube) (unit cube) is 1. Haar-Lebesgue measure in $\mathcal{I}^m \times \mathcal{R}^p$, denoted by μ, is the product of σ_0 and σ_1. Let $A \subset \mathcal{R}^q$. We write μA in place of $\mu(A \cap (\mathcal{I}^m \times \mathcal{R}^p))$. If $x \in \mathcal{R}^q$, then $\sigma(A, x)$ denotes $\sigma(A \cap (A + x))$ and $\mu(A, x)$ denotes $\mu(A \cap (A + x))$. If n is a positive integer, then nA denotes $[a_1 + \cdots + a_n \mid a_i \in A_i\,;\ i = 1, \cdots, n]$ and $n^{-1}A$ denotes $[n^{-1}a \mid a \in A]$. If f and g are functions of a positive integer n, then $f_n \sim g_n$ shall mean $\lim_{n\to\infty} f_n/g_n = 1$. We note that $\sim$ is an equivalence relation.

Lemma 1. *Let A be a compact convex subset of $\mathcal{R}^q$ with* int $A \neq \phi$ *and let $x \in \mathcal{R}^q$. Then $\mu nA \sim \mu(nA, x) \sim \sigma(nA, x) \sim \sigma nA$.*

Proof. We show $\mu nA \sim \sigma nA$. For each $y = (y_1, \cdots, y_q) \in \mathcal{I}^q$ we define the unit cube $B_y = [z \mid y_i \leq z_i < y_i + 1;\ i = 1, \cdots, q] \subset \mathcal{R}^q$. We observe

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