SELF-DUAL POSTULATES FOR n-DIMENSIONAL GEOMETRY

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1. **Introduction.** One of the outstanding properties of projective geometry is the duality of points and lines in two dimensions, and of points and planes in three dimensions. Although we all know this duality, and feel that it should be fundamental, we still keep non-dual concepts of points and planes, points (and lines) being primordial, while planes are constructed by generation of lines. In this article, however, we shall take points and planes as primordial, and define lines as particular sets of points and planes. The classical axioms of projective geometry are non-dual (see for instance [4; vol.I, 16, 24]). We believe, with Karl Menger, that for greater unity and continuity of thought, the axioms should already show this duality, and therefore we propose the four postulates Ia, Ib, II (§2) and III''' (§5) for three dimensions. (Self-dual postulates have been introduced by Karl Menger, but we believe that our article introduces additional simplifications, in particular concerning lines, which we define while Menger considers them as primitive notions, therefore needing more postulates.) As little is gained from restricting the number of dimensions to three, we have treated the general case of n dimensions, introducing then the five postulates of §2.

Points and planes will be studied first (§§2, 3), then lines and flats will be defined and investigated (§4). We conclude with some comments on the postulates. We shall consider only a minimum set of postulates, which gives a very unrestricted, and yet non-trivial geometry. The additional postulates which are usually introduced later (see for instance [4; vol.I, 18, 45, 95; vol.II, 3, 11, 16, 23, 32, 33]) can be proved to be equivalent to their own duals.

Whenever two mutually dual statements are made, the first one will be denoted by "a" and the second by "b". Only statement a will be proved.

2. **The postulates.** We start with undefined elements called *points* and *planes* (more exactly *hyperplanes*), and with an undefined *on* relation enabling us to distinguish a point and a plane on each other from a point and a plane not on each other.

Definition I (*of a simplex*). A set of $n + 1$ points $A_0, \cdots, A_n$ and $n + 1$ planes $a_0, \cdots, a_n$ such that A_i is on a_j if $i \neq j$, and A_i is not on a_i, is called a (n-dimensional) *simplex*. The points $A_0, \cdots, A_n$ are called the *vertices*, and the planes $a_0, \cdots, a_n$ are called the *faces*.

The number n shall be fixed throughout our discussion, and all our simplexes shall be n-dimensional. By the simplex (P, p) we shall mean the simplex whose

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