

CONVEX SETS IN LINEAR SPACES

BY V. L. KLEE, JR.

1. **Introduction.** The object of this paper is to establish certain properties of convex sets in a setting as general as possible. This generality is perhaps interesting in itself since it sometimes leads to unusually simple proofs. Furthermore, this continues the work of Bourgin [5] in bringing development of the theory of convex sets in linear spaces more nearly abreast with that of functional analysis in general. For example, the study of Banach spaces in their various weak topologies and of Hausdorff linear spaces in general becomes increasingly important, but certain fundamental separation and support theorems for convex sets have not previously been established in such spaces. Our principal theorems extend results of Bourgin, LaSalle, Mazur, Schoenberg, Tukey, and the author.

It is to be expected that some of the results presented here as new to the author are more or less widely known, although they seem never to have appeared in print. (For example, the author was unaware of the existence of the "ubiquitous" convex sets described in §8 until they were mentioned in a referee's report on one of his papers as the "standard" example showing that certain properties of convex sets do not persist in infinite-dimensional spaces.) However, their fundamental nature should justify recording them here.

As an indication of contents we list the headings of the remaining sections:

2. Some definitions and preliminary observations.
3. The core topologies of linear systems.
4. Some basic properties of convex sets.
5. Local convexity.
6. Relations between the core and the interior of a set.
7. Maximal varieties and hyperplanes.
8. Some properties of linear systems.
9. Separation and support properties.
10. A conjecture of Bourgin.
11. Non-support points of convex sets.
12. Results on p -functionals.

There are numerous examples, questions, and suggestions for further research.

The contents of this paper were presented to the American Mathematical Society under six different titles, it being originally intended that these should appear separately as a series of notes. However, it became apparent that this would entail a considerable amount of repetition and accordingly it was decided to combine them. Two papers, already accepted for publication, were withdrawn for this purpose from the Bulletin of the American Mathematical Society.

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