RINGS WITH IDEAL NUCLEI

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1. **Introduction.** In this paper we analyze and generalize decomposition theorems for certain topological rings obtained by van Dantzig [3] and Kaplansky [6]. Both these authors were ultimately concerned with rings whose neighborhoods of zero were ideals. Our analysis of the situation essentially consists in the statement that a complete ring with ideal neighborhoods of zero decomposes into a direct sum when (and only when) the residue class rings modulo these open ideals decompose concordantly (Theorem 1).

Such a concordant decomposition is available when the ring is commutative and has a restricted minimum condition on open ideals, so that the residue rings are commutative with minimum condition (§4; Theorem 5). The original ring decomposes into summands which are something like $\mathfrak{p}$-adic completions of the ring. This is much like van Dantzig's situation but without the hypotheses of an identity element, maximal condition, and countability. It should be remarked, however, that the condition van Dantzig uses in place of the restricted minimum condition, while more general, still fits into our pattern of giving a decomposition theorem because of decomposition of the discrete residue rings. Kaplansky's theorem on compact commutative rings [6; Theorem 17] is a special case of our theorem, at least in the totally disconnected case where ideal neighborhoods of zero with finite residue rings are at hand.

Another instance of decomposition of residue rings occurs when the latter are semisimple with minimum condition. This is a "concordant" decomposition when the original ring is semisimple and satisfies a minimum condition modulo open ideals. A complete ring of this type is a complete direct sum of classical simple rings (§5; Theorem 7).

In §6, the results of §4 are applied to compute the completions of certain topological fields first studied in [9] having as a nuclear base the non-zero ideals of a subring. The topologies thus obtainable for algebraic number fields are shown to be not much more general than the ordinary $\mathfrak{a}$-adic topologies and give completions which are local direct sums (direct sums in the $\mathfrak{a}$-adic case) of $\mathfrak{p}$-adic completions of the field.

In §§2 and 3, we justify the statement that decompositions of our rings are due to decompositions of residue class rings. The machinery is founded on the concept of inverse limits of rings, which provides a convenient tool (every complete ring with ideal neighborhoods of zero is the inverse limit of its discrete residue rings). This machinery need not have been limited to rings with ideal neighborhoods of zero. We could have considered rings with right ideal neighborhoods of zero, topological groups with operators with admissible normal

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