PSEUDO-CONFORMAL TRANSFORMATIONS ONTO CIRCULAR DOMAINS

By George Springer

Introduction. Let D be a domain in the (z_1, z_2)-space, where z_1 and z_2 are each complex variables, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. A mapping of D onto a domain B of the (w_1, w_2)-space, $w_1 = u + iv_1$, $w_2 = u_2 + iv_2$; i.e., $w_1 = w_1(z_1, z_2)$, $w_2 = w_2(z_1, z_2)$ is called pseudo-conformal if w_1 and w_2 are each analytic functions of the two variables z_1 and z_2 and if their functional determinant $\partial(w_1, w_2)/\partial(z_1, z_2)$ does not vanish in D.

In the conformal mapping of plane regions, the Riemann mapping theorem allows us to map any simply connected region with at least two boundary points onto the unit circle. No such theorem exists in the space of two complex variables. It has already been demonstrated that an infinite number of different canonical domains are necessary in order to be able to map any simply connected domain onto one of our canonical domains, but no such system of canonical domains has yet been determined.

One of the basic problems in the theory of pseudo-conformal mapping is the determination of when a given domain can be mapped pseudo-conformally onto a certain canonical domain. In this paper, we consider the problem of determining when a given domain can be mapped pseudo-conformally onto a representative of a certain class of circular domains, which are a natural generalization of the unit circle. Closely connected with the mappings onto canonical domains are the various theorems on the distortion under pseudo-conformal mapping. Even though the given domain D may be very complicated, certain properties of the mapping of D onto a canonical domain can be determined if we can find simple domains for which certain basic quantities can be computed explicitly and which either contain D or are contained in D. Such simpler domains are called domains of comparison and are used to yield various distortion theorems. The methods used depend largely upon the theory of orthogonal functions and the kernel function developed by S. Bergman [1], [2].

In the first section, the Bergman kernel and an associated pseudo-conformal invariant are determined for the domain, $E(\lambda)\colon |z_2|^2 < \exp(-\lambda |z_1|^2)$, which can in general be used as an exterior domain of comparison, and also for certain product domains that can be used as interior domains of comparison. These lead to various distortion theorems, an example of which is given for the mapping of a domain contained between $E(\mu)$ and $E(\nu)$ onto the domain $E(\lambda)$.

§2 is devoted to the complete treatment of the problem of determining whether a given simply connected domain D can be mapped onto a circular

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