SUMS OF TWO SQUARES IN A QUADRATIC FIELD

By Gordon Pall

1. **Introduction.** Niven [3] has found necessary and sufficient conditions for an integer in a quadratic field to be representable as a sum of two (or more) squares of integers in the field. His approach is based on a theorem due to Mordell [2] giving necessary and sufficient conditions for a binary quadratic form to be expressible as a sum of two (or more) squares of linear forms with integer coefficients.

While the *number* of representations of a binary quadratic form as the sum of n squares of linear forms has been studied (notably by Hel Braun [1] for the case of odd determinant and $2 \leq n \leq 8$; and by the author [4], [5] quite generally for $n = 3$), a remarkably simple form which it can be given in the case $n = 2$ has been overlooked. Indeed, let $r_2(d)$ denote the number of representations of d as the sum of squares of two ordinary integers. In order that the binary quadratic form ϕ be expressible as a sum of two squares of linear forms with integral coefficients, it is necessary that ϕ shall be positive definite or semidefinite, have an integral matrix, and have a square determinant. If these conditions are satisfied, the number of all such representations (given by Theorem 1) is equal to $r_2(d_1)$ or $2r_2(d_1)$, where d_1 denotes the divisor of ϕ, according as the determinant of ϕ is or is not zero.

This result makes it possible to derive simple formulae for the number of representations as a sum of two squares in a quadratic field. The examples of rings $a + b\rho$, where $\rho^2 = -1, 1, 2$, or $\rho^2 + \rho - 1 = 0$, are given in detail. Other cases, and the extension to forms other than a sum of two squares, will be investigated by students of the author.

The number of representations as a sum of two squares is obviously finite in every real quadratic field. It is shown in §3 that this number is infinite (when not zero) in every imaginary quadratic field with the sole exception of that in which $\rho^2 = -1$. In this ring, of Gauss integers $a + bi$ (a and b ordinary integers, $i^2 = -1$), the number of representations as a sum of two squares $(a_1 + b_1 i)^2 + (a_2 + b_2 i)^2$ is always finite (and given by Theorem 2), save for the trivial case of $0 + 0i$. By defining *sets* of representations appropriately (by use of unimodular automorphs, or, if preferred, of the representations of 1), it would no doubt be possible to prove similar formulae giving a finite number of sets of representations in other imaginary quadratic fields.

The methods of this article suggest the existence of a connection between the class numbers of binary quadratic forms (with ordinary integer coefficients) of determinants d and $-d$, and of binary quadratic forms of determinant 1 with coefficients in the quadratic field generated by $d^{\frac{1}{2}}$.

Received July 25, 1949.