THE VOLUME OF A CERTAIN MATRIC DOMAIN

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Introduction. Let $A < B$ mean that the real symmetric matrix $B - A$ is positive definite and $A \leq B$ mean that $B - A$ is positive semi-definite. Suppose that we are given two real symmetric positive definite matrices S, T of order m, n respectively. Let X be a real $m \times n$ variable matrix and $\mathfrak{E}(S, T)$ be the domain in the mn dimensional space of the variable X defined by $X'SX < T$, where X' denotes the transpose of X. We shall find the Euclidean volume $h(S, T)$ of the domain $\mathfrak{E}(S, T)$.

The domain $X'SX < T$.

1. Let

$$\{dX\} = \prod_{i=1}^{m} \prod_{j=1}^{n} dx_{ij}$$

and

$$h(S, T, \alpha) = \int_{\mathfrak{E}(S,T)} | T - X'SX |^{\alpha} \{dX\} \tag{1}$$

so that the required volume $h(S, T) = h(S, T, 0)$. We shall write

$$h_{m,n}(\alpha) = h(1_m , 1_n , \alpha), \qquad h_{m,n} = h_{m,n}(0), \tag{2}$$

where 1_n denotes the unit matrix of order n. Also we put $\mathfrak{E}_{m,n} = \mathfrak{E}(1_m , 1_n)$ so that $\mathfrak{E}_{m,n}$ denotes the mn dimensional "unit sphere", $X'X < 1_n$.

Theorem 1.

$$h(S, T, \alpha) = | S |^{-n/2} | T |^{\alpha + m/2} h_{m,n}(\alpha).$$

Proof. Determine positive definite matrices S_1, T_1 such that $S = S_1^2$, $T = T_1^2$, and put $Y = S_1 X T_1^{-1}$ in (1). Then $\mathfrak{E}(S, T)$ maps into $\mathfrak{E}_{m,n}$ and

$$\{dX\} = | S_1 |^{-n} | T_1 |^{m} \{dY\} = | S |^{-n/2} | T |^{m/2} \{dY\}$$

so that

$$h(S, T, \alpha) = | S |^{-n/2} | T |^{\alpha + m/2} \int_{\mathfrak{E}_{m,n}} | 1_n - Y'Y |^{\alpha} \{dY\}.$$

We shall next prove a generalization of Schmidt's theorem (see [5; 96]).

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