REMARKS ON THE ISOPERIMETRIC INEQUALITY

BY R. G. HELSEL

1. **Introduction.** The usual attack on the problem of establishing the spatial isoperimetric inequality in terms of the Lebesgue area involves two steps. First, the inequality is proved for polyhedra; that is, if P is any closed polyhedron with elementary area $A(P)$ and enclosed volume $V(P)$, it is shown that

$$V(P)^2 \leq \frac{A(P)^3}{36\pi}. \tag{1.1}$$

Second, the inequality is obtained for a general closed surface S by using the fundamental fact that there exists at least one sequence of polyhedra P_n which converges in area to S; in other words, the polyhedra converge to S and their elementary areas also converge to the Lebesgue area of S. Since (1.1) holds for each polyhedron P_n and the sequence $A(P_n)$ converges to the Lebesgue area $A(S)$ of S, the general isoperimetric inequality

$$V(S)^2 \leq \frac{A(S)^3}{36\pi} \tag{1.2}$$

follows providing the sequence $V(P_n)$ converges to the enclosed volume $V(S)$ of S or, failing that, if the limit inferior or limit superior of the sequence $V(P_n)$ is not less than $V(S)$.

None of the early papers [2], [3], [9] on the isoperimetric inequality contains a formal definition for the enclosed volume $V(S)$ and yet all the authors assume that if a sequence of polyhedra P_n converges in area to a limit surface S then the sequence of enclosed volumes $V(P_n)$ converges to the enclosed volume $V(S)$. Actually not one of the several plausible definitions for enclosed volume enjoys this convergence property (see [6] for further discussion of this point). However, the enclosed volume introduced by Radó [6],

$$V_R(S) = \iiint |\, i(x, y, z; S) \,|\, dx\, dy\, dz \tag{1.3}$$

if $i(x, y, z; S)$ is summable, $V_R(S) = +\infty$ otherwise, where $i(x, y, z; S)$ is the topological index of the point (x, y, z) with respect to the oriented closed surface S and the integral is taken over all of xyz-space, has been shown [4] to possess the aforementioned convergence property provided the limit surface occupies a point set of zero three-dimensional Lebesgue measure. In addition, $V_R(S)$ is a lower semi-continuous functional with respect to a convergent sequence of closed surfaces (see [6]). Thus the general isoperimetric inequality (1.2) in

Received September 27, 1949.