SECOND ORDER DETERMINANTS OF LEGENDRE POLYNOMIALS

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1. Introduction and summary of results. For each non-negative integer r let $P_r(x)$ be the Legendre polynomial of r-th degree on the interval $(-1, 1)$, normalized, as in [5; Chapter XV], so that $P_r(1) = 1$. For integers n, h, k such that

$$n \geq 0, \qquad k \geq h \geq 1, \tag{1}$$

we define the following function of the real variable x:

$$\Delta = \Delta(n, h, k; x) = \begin{vmatrix} P_n(x) & P_{n+h}(x) \\ P_{n+k}(x) & P_{n+h+k}(x) \end{vmatrix}.$$

When it is not specified otherwise, it will be assumed that n, h, k are integers satisfying (1). When $h + k$ is an even [odd] integer, Δ is an even [odd] function of x. Clearly $\Delta(n, h, k; 1) = 0$.

Definition. Let n, h, k be given. The determinant $\Delta = \Delta(n, h, k; x)$ is said to have property T when $0 < x < 1$ implies that $\Delta < 0$. (The T is for Turán.)

The general purpose of this investigation is to see which of the determinants Δ have property T. Turán discovered that $\Delta(n, 1, 1; x)$ has property T for all $n \geq 0$, and Szegö gave several proofs of this in [3]. In §2 and §3 it will be shown that $\Delta(n, 1, 2; x)$ and $\Delta(2n + 1, 2, 2; x)$ both have property T for all $n \geq 0$. The proofs involve applying Szegö's first method of proof to Δ, $d\Delta/dx$, or $d^2\Delta/dx^2$ in various subintervals of the interval $(0, 1)$. The inequality for $\Delta(2n + 1, 2, 2; x)$ is shown in §3 to be equivalent to an inequality of the original Turán type for the Jacobi polynomials $P_n^{(0, \frac{1}{2})}(x)$ in the notation of [4; Chapter 4].

On the other hand, the table on p. 362 summarizes the triples (n, h, k) for which it is proved in §4 that $\Delta(n, h, k; x)$ fails to have property T. The Roman numerals refer to the cases within §4.

The triples of the table have an asymptotic density of seven-eighths within the class of triples (n, h, k) satisfying (1). One wonders whether $\Delta(n, 1, 1; x)$, $\Delta(n, 1, 2; x)$, $\Delta(2n + 1, 2, 2; x)$ may be the only determinants of type $\Delta(n, h, k; x)$ with property T.

In a related paper [1] dealing with determinants of higher orders, Beckenbach, Seidel and Szász have shown (as a special case of their Theorem 4) for all n,

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