NOTE ON AN INFINITE INTEGRAL

By Alexander M. Ostrowski

In this note we prove (Theorem II), that from the convergence of an integral

$$\int_p^\infty \frac{f(ax) - f(bx)}{x^{\alpha+1}}\, dx \qquad (p, a, b, \alpha > 0)$$

follows the convergence of the integral $\int_p^\infty f(x)x^{-(\alpha+1)}\, dx$. That this is no longer true for $\alpha = 0$ is well known and is the essential point behind the so-called Frullani theorem.

This result appears to be new while our Theorem I, the relation

$$\int_0^\infty \frac{f'(x)}{x^\alpha}\, dx = \alpha \int_0^\infty \frac{f(x) - f(+0)}{x^{\alpha+1}}\, dx \qquad (\alpha > 0),$$

is, in the case $\alpha = 1$, more or less old, since a formula by Winckler,

$$\text{(W)} \qquad \int_0^\infty \frac{f(bx) - f(ax)}{x^2}\, dx = (b - a) \int_0^\infty \frac{f'(x)}{x}\, dx$$

is an immediate corollary of our formula for $\alpha = 1$. (The integral on the left in (W) has been considered by J. Bertrand [1; 225] and G. Frullani [2; 462]. However, the results given by both authors are not correct.) I have been unable to find in the literature the Theorem I as I prove it.

Lemma I. *Let for a positive p and a positive α the integral*

$$(1) \qquad \int_0^p \frac{\varphi(t)}{t^\alpha}\, dt = \lim_{\epsilon \downarrow 0} \int_0^p \frac{\varphi(t)}{t^\alpha}\, dt$$

exist. Then the integral

$$(2) \qquad \int_0^p \varphi(t)\, dt = \lim_{\epsilon \downarrow 0} \int_\epsilon^p \varphi(t)\, dt$$

exists, and we have

$$(3) \qquad \left(\frac{1}{x}\right)^\alpha \int_0^x \varphi(t)\, dt \to 0 \qquad (x \downarrow 0).$$

(In (1), (2), (5), (9) and (13), the right-hand integral is to be understood as a Lebesgue integral.)

Proof. For $0 < x_0 < x < p$ we have by the second mean value theorem (for this theorem in the case of Lebesgue integrals, see [3; 231]) the relation

$$(4) \qquad \int_{x_0}^x \varphi(t)\, dt = x_0^\alpha \int_{x_0}^\xi \frac{\varphi(t)}{t^\alpha}\, dt + x^\alpha \int_\xi^x \frac{\varphi(t)}{t^\alpha}\, dt,$$

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