SOME STUDIES ON CYCLIC DETERMINANTS

BY OYSTEIN ORE

A determinant is said to be *cyclic* when it has the form

$$(1) \qquad \mathfrak{C}(a_0, \cdots, a_{n-1}) = | a_{j-i} | \quad (i, j = 0, \cdots, n-1),$$

where $a_r = a_s$ for $r \equiv s \pmod{n}$. These determinants appear in a number of questions in algebra and number theory and there exists a rather considerable literature on the subject. (See, for instance, Muir [3] and [4].) In the following we shall deduce various properties which I have been unable to find elsewhere.

1. **Explicit form of the cyclic determinants.** To determine the explicit form of the expansion of a cyclic determinant (1) we take as our starting point the well-known identity

$$(2) \qquad \mathfrak{C}(a_0, \cdots, a_{n-1}) = \prod_{i=1}^{n} (a_0 + a_1\alpha_i + a_2\alpha_i^2 + \cdots + a_{n-1}\alpha_i^{n-1}),$$

where the α_i run through the n-th roots of unity. When the multiplication in (2) is executed one finds

$$(3) \qquad \mathfrak{C}(a_0, \cdots, a_{n-1}) = \sum \phi_{p_1 \cdots p_r} a_0^{n-r} a_{p_1} \cdots a_{p_r},$$

where the indices p_i belong to the set $1, \cdots, n-1$ and may take equal values. To each term

$$(4) \qquad a_0^{n-r} a_{p_1} \cdots a_{p_r}$$

there is associated a *weight*

$$(5) \qquad P = p_1 + \cdots + p_r .$$

Often it is convenient to write the term (4) in the slightly different form

$$(6) \qquad a_0^{\nu_0} a_1^{\nu_1} \cdots a_{n-1}^{\nu_{n-1}},$$

where $\nu_0 = n - r$ and

$$(7) \qquad \nu_0 + \nu_1 + \cdots + \nu_{n-1} = n, \qquad P = \nu_1 + 2\nu_2 + \cdots + (n-1)\nu_{n-1} .$$

Our first problem is to determine the rational integral coefficients $\phi_{p_1 \cdots p_r}$ which appear in the expansion (3). From (2) we conclude that they are symmetric functions of the n-th roots of unity

$$(8) \qquad \phi_{p_1 \cdots p_r} = \sum \alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_r^{p_r}.$$

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