ENDOMORPHISMS OF LATTICES

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It is the object of the present investigation to develop a theory of homomorphisms for lattices by generalizing the concept of homomorphism as used in group theory. In addition to a derivation of the basic isomorphism laws, the paper gives a generalization of Fitting's lemma and of the theory of splitting endomorphisms.

The main results on uniformly splitting endomorphisms of lattices are established by the introduction of the concept of an η-automorphic element. It is shown that a necessary condition for uniform splitting is that the sum of η-automorphic elements be η-automorphic and that under the assumption of uniform splitting the complement is uniquely determined as the sum of all η-automorphic elements. There are also included reduction theorems and necessary and sufficient conditions for uniform splitting.

These results on uniform splitting are useful in obtaining a generalization of Fitting's method of investigating the refinement theorems for groups or loops. It is the purpose of a later paper to generalize theorems in the theory of direct decompositions by translating Fitting's method into lattice-theoretical concepts. Material in this paper was contained in a doctoral thesis written under the direction of Professor Reinhold Baer. The author wishes to take this opportunity to express her appreciation to Professor Baer for his suggestions and aid.

I. The Fundamental Concepts

In this part we state the postulates and basic definitions and develop fundamental results on homomorphisms and endomorphisms.

1. **The Postulates.** The algebraic system to be considered is a complete modular lattice P upon which is imposed one additional postulate. The elements of P are denoted by lower-case italic letters and the partial ordering by $\leq$. If p and q are elements in P, there exist (uniquely determined) elements pq, the *product* of p and q, and $p + q$, the *sum* of p and q, such that $pq \leq p \leq p + q$, $pq \leq q \leq p + q$, and $b \leq p \leq a$ and $b \leq q \leq a$ together imply that $b \leq pq$ and $p + q \leq a$. In addition to Dedekind's modular law, we assume also the existence of an element 0 in P such that $0 \leq p$ for every p in P. The last condition imposes no essential loss of generality, since without it only those elements in P which contain a suitable element would be considered. The following additional property is assumed:

Postulate (*). *If a and $\{p_r\}$, where r ranges over a set of ordinals, are elements in P and if the p_r form an ascending chain, then* $a \sum_r p_r = \sum_r ap_r$.

Received June 9, 1949; in revised form, December 22, 1950.