PROBABILITY IN DYNAMICAL TRANSFORMATION GROUPS

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1. **Introduction.** In the study of classical dynamical systems, it has proved useful to regard such a system as a one-parameter transformation group acting on a topological space. A quite natural generalization of this is to drop the restriction that the group be a one-parameter group and instead consider a topological group of transformations acting on a topological space. In recent papers Barbachine, Gottschalk, Niemytski, Krasnosel'skiĭ and Kreĭn, and Gottschalk and Hedlund have studied this type of generalization. In this paper we study this general type of system from the point of view of probability.

Among the notable contributions to the study of classical dynamical systems from the point of view of probability are those of G. D. Birkhoff (see [3; Chapter VII] and [6]). As Hilmy's results are more precise than those of Birkhoff, we have followed Hilmy's approach to the problem.

In §2, a definition of probability is given and its fundamental properties developed. We show that if the group is the additive group of reals or integers with their natural topology then our definition of probability reduces to the classical definition. In §§3 and 4 we prove theorems which generalize those of Hilmy and Birkhoff. We also prove a theorem (Theorem 4.3) which does not appear in the works of Hilmy or Birkhoff, probably because the proof is trivial in the classical system, though by no means trivial in the more general system.

2. **Definition and properties of probability.** Let T be a multiplicative Abelian topological group which is separable, metric, locally compact and has a compact generating system. Recall that a topological group, T, is said to have a compact generating system if there exists an open neighborhood of the identity whose closure, K, is compact and such that $T = \bigcup_1^\infty K^n$ where K^n consists of all possible products $t_1 t_2 \cdots t_n$, $t_i \in K$, $(i = 1, 2, \cdots, n)$. Let μ be the Haar measure on T. Let X be a topological space and let T act as a transformation group on X; that is to say, suppose that to each $x \in X$ and $t \in T$ is assigned a point, denoted xt, of X such that: 1) $xe = x$ for all $x \in X$ where e is the identity in T; 2) $(xt)s = x(ts)$ for all $x \in X$ and all $t, s \in T$; 3) the function xt defines a continuous transformation of $X \times T$ onto X.

Definition 2.1. Let A be a subset of X and V a subset of T. Let $x \in X$. The set of all $t \in V$ such that $xt \in A$ will be denoted by $T(x, A, V)$. Let s be an arbitrary element of T. Let W be an open generator of T which contains the identity, e, and such that $K = {}^c(Ws)$ is compact. (${}^c(Ws)$ means the closure of Ws.) If $\mu T(x, A, K^n)$ exists for all choices of K and for all positive integers n and if

$$\lim_{n\to\infty} \frac{\mu T(x, A, K^n)}{\mu(K^n)}$$

Received June 6, 1949; in revised form, October 14, 1949.