A COMPOSITION THEOREM FOR ASYMPTOTIC SERIES

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It is well known that if $f(t)$ is infinitely differentiable on an interval containing the origin, if $| f^{(n)}(t) | < M_n$, $n \geq 0$, with $\log M_n$ a convex function of n, if $f^{(n)}(0) = 0$, $n \geq 0$, and if $\sum M_n/M_{n+1} = \infty$, then $f(t) \equiv 0$. Mandelbrojt has recently given [1] similar conditions assuring that if, for each non-negative integer k, the $2k$-th derivative of at least one of two even functions, infinitely differentiable for all values of the argument, is zero at the origin, then one at least of the two functions vanishes identically. This result is given in the following theorem.

Theorem 1. *If*

(1) $f(t)$, $\phi(t)$ *are even functions infinitely differentiable on* $(-\infty, \infty)$;

(2) $| f^{(n)}(t) | < M_n$, $n \geq 0$, $\int_0^t f(\tau)\, d\tau = O(1)$ *as* $| t | \to \infty$;

$| \phi^{(n)}(t) | < M'_n$, $n \geq 0$, $\int_0^t \phi(\tau)\, d\tau = O(1)$ *as* $| t | \to \infty$;

(3) $f^{(2n)}(0)\phi^{(2n)}(0) = 0$, $n \geq 0$;

(4) $\sum M_n M'_n / M_{n+1} M'_{n+1} = \infty$, *where* $\log M_n$ *and* $\log M'_n$ *are convex functions of* n;

then either $f \equiv 0$ *or* $\phi \equiv 0$.

More generally, Mandelbrojt has proved that if $f_1(t), \cdots, f_N(t)$ are even, infinitely differentiable functions; if $| f_k^{(n)}(t) | < M_n^{(k)}$, $k = 1, \cdots, N$, $\log M_n^{(k)}$ being convex in n; if $\int_0^t f_k(\tau)\, d\tau = O(1)$, $k = 1, \cdots, N$, as $| t | \to \infty$; if $\prod_{k=1}^N f_k^{(n)}(0) = 0$; if $\sum M_n^{(1)} \cdots M_n^{(N)} / M_{n+1}^{(1)} \cdots M_{n+1}^{(N)} = \infty$; then one of the functions $f_k(t)$ is identically zero. The proof is similar to that of the theorem mentioned above and given in [1].

Frequent use has been made of the correspondence given by the Laplace transform between functions infinitely differentiable on $[0, \infty)$ having bounded derivatives, and functions of a complex variable having asymptotic representations. If $f(t)$ is infinitely differentiable on $[0, \infty)$, and if $| f^{(n)}(t) | < M_n$, repeated integration by parts shows that the function $F(z) = \int_0^\infty e^{-zt} f(t)\, dt$ has an asymptotic representation with coefficients $\{f^{(k-1)}(0)\}$ with respect to the sequence of constants $\{M_n\}$ in a certain half-plane: $| F(z) - \sum_1^n$

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