SURFACE INTEGRALS OF THE WEIERSTRASS TYPE

By George M. Ewing

1. **Introduction.** It is customary in mathematical physics to associate with each elementary surface a vector of magnitude equal to the area, with its direction and sense determined by the normal and the rule of the right-hand screw. The surface integral is then a limit of a suitable sum, attention being restricted to smooth or at worst piece-wise smooth surfaces.

The object of this paper is to study integrals over surfaces of finite Lebesgue area, starting with an adaptation of the classical procedure. Cesari has defined [1; 81] such a surface integral as a certain triple limit using an arbitrary representation. Our single limit, introduced in §2, demands a representation with particular properties but which fortunately is always available. Our integral seems to be simpler than that of Cesari or the corresponding integrals of Lebesgue type employed in the recent literature. The largely axiomatic procedure is novel. It leads rather directly to the principal theorems and provides a pattern which suggests the possibility of extension to other cases.

The writer has depended heavily upon Rado [5] and [6] and Cesari [1]. He is indebted to P. V. Reichelderfer for criticism of an early draft and to the referee who discovered errors in §1. He had the benefit of communications with L. Cesari, E. J. Mickle and J. W. T. Youngs concerning the result of Cesari used in §5.

Surface will mean oriented Fréchet surface of the type of the 2-cell [5; 342]. A *representation* of a surface is a vector-function $x(u)$ with components $x^i(u)$, $i = 1, 2, 3$, the parameter $u \equiv (u^1, u^2)$ ranging over a Jordan region R. When R is polygonal it is denoted alternatively by P.

A *partition* π of R is a finite set of non-overlapping Jordan sub-regions R_j, $j = 1, 2, \cdots, k$, whose sum is R. Each R_j is called a *piece* of R. The norm $n(\pi)$ is the sup of the diameters of the pieces of R which constitute π.

The symbol $S : x(u),\ u \varepsilon R$, is to be read "a (or the) surface S having $x(u)$, $u \varepsilon R$, as a representation". Given a partition π, $S_j : x(u),\ u \varepsilon R_j$, is called a piece of S. The set of all such pieces S_j, $j = 1, 2, \cdots, k$, is a *decomposition of* S *relative to* $x(u)$, $u \varepsilon R$, *and to* π.

The set in x-space onto which $x(u)$ maps R will be denoted by *graph* S.

Fréchet distance [5; 341, 342] between representations $x(u)$ and $y(v)$ or between surfaces S and T is denoted respectively by $\delta(x, y)$ or $\delta(S, T)$. That S is *oriented* means that homeomorphisms $H : u = h(v)$ in the definition of distance are sense-preserving.

Received July 8, 1949; in revised form, April 17, 1950 and May 4, 1950. Preliminary report presented to the American Mathematical Society, April 30, 1949. Completed with the aid of a Summer Research Professorship at the University of Missouri during the Summer Session of 1949.