THE COMPARISON OF SPECTRA BELONGING TO POTENTIALS WITH A BOUNDED DIFFERENCE

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1. Let λ denote a real parameter and let $p(t)$, $q_1(t)$ and $q_2(t)$ be real-valued, continuous functions on the half-line $0 \leq t < \infty$ satisfying

$$p(t) > 0, \ | q_1(t) - q_2(t) | < \text{constant} < \infty \qquad (0 \leq t < \infty). \tag{1}$$

It is known that either both or neither of the differential equations

$$(px')' + (\lambda + q_k(t))x = 0 \qquad (k = 1, 2), \tag{2_k}$$

is (in the terminology of Weyl [6; 238]) in the Grenzpunktfall. In fact, Wintner has observed this fact to be a corollary of a more general theorem concerning integral kernels due to Carleman [1; 72, 78]. For if the two expressions appearing on the left of each of the equations (2_k) are denoted by $L_k(x)$, where L_k, $k = 1, 2$, are differential operators, it is seen that $L_1 - L_2 = q_1 - q_2$; hence, by (1), this difference is a bounded operator in the Hilbert's space $L^2[0, \infty)$. The desired result then follows from [1; 78], where it is shown that the "class" of an integral kernel is unchanged by adding to it a bounded kernel. If both equations (2_k) are in the Grenzpunktfall, let $S_k(\alpha)$ denote the spectrum of the boundary value problem on $0 \leq t < \infty$ determined by (2_k) and the boundary condition

$$x(0) \cos \alpha + x'(0) \sin \alpha = 0 \qquad (0 \leq \alpha < \pi). \tag{3}$$

The set of cluster points of the set $S_k(\alpha)$ is independent of α [6; 251] and will be denoted by S'_k, $k = 1, 2$. The following will be proved:

(*) *Let $p(t)$, $q_1(t)$ and $q_2(t)$ denote continuous functions on $0 \leq t < \infty$ satisfying* (1) *and such that both equations* (2_k), $k = 1, 2$, *are in the Grenzpunktfall.* (i) *If α is fixed and μ_1 denotes any point of the set $S_1(\alpha)$, then there exists at least one point μ_2 of the set $S_2(\alpha)$ satisfying*

$$| \mu_1 - \mu_2 | \leq \beta, \qquad \textit{where } \beta = \operatorname*{l.u.b.}_{0 \leq t < \infty} | q_1(t) - q_2(t) |. \tag{4}$$

(ii) *If μ_1 denotes any point of the set S'_1, there exists at least one point of the set S'_2 satisfying*

$$| \mu_1 - \mu_2 | \leq \gamma, \qquad \textit{where } \gamma = \limsup_{t \to \infty} | q_1(t) - q_2(t) |. \tag{5}$$

If $p(t) \equiv 1$, $| q_1(t) | <$ constant and $q_2(t) \equiv 0$, the theorem of [4] readily follows from (*).

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