A CLASS OF OPERATOR ALGEBRAS WHICH ARE DETERMINED BY GROUPS

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1. **Introduction.** In the present paper we define and treat two classes of mathematical models for the system of mechanical observables associated with an elementary particle, these models consisting of the aggregates of all self-adjoint elements in certain algebras of bounded operators on Hilbert space. Special cases of these models represent the usual relativistic and non-relativistic theories, and we give detailed consideration to an interesting model for which the analogue of space is discrete and which has the relativistic model as a kind of limiting case. Although the algebras of operators which we consider contain bounded operators exclusively, the identification of operators with physical quantities, and also certain mathematical problems, are facilitated by the utilization of unbounded operators. For this reason we have investigated the generators of unitary representations of Lie groups, most of the unbounded operators occurring in quantum mechanics being functions of such generators, and shown that these operators, as well as certain formally self-adjoint polynomials in them, are essentially hypermaximal symmetric or, in rough terms, have unique diagonal forms. This result permits us to define, in a form which is from a technical viewpoint relatively simple, the values in an arbitrary state of the system of the corresponding observables.

A model in the first class is determined by a transformation group acting on a space with an invariant measure. A wave function, i.e., a square integrable function over the space of unit norm, will correspond to a pure state if and only if the group is ergodic (but, as is known, not all pure states correspond to wave functions). The usual theories are of this type, the relativistic and non-relativistic theories corresponding to the models obtained from the inhomogeneous Lorentz group and the group of three-dimensional Euclidean space acting on pseudo-Euclidean space-time and three-dimensional Euclidean space, respectively, with Lebesgue measure in either case. A model in the second class is determined by a locally compact group and is a variety of group algebra of the group. In general, wave functions will not correspond to pure states, and to determine the pure states, one of the most practicable means appears to be to determine the irreducible representations on Hilbert space of the Lie algebra of the group. The model mentioned above as being considered in some detail is for the case in which the group is taken as the covering group of the conformal group of pseudo-Euclidean space-time which is a 15-dimensional simple

Received July 3, 1948; in revised form, November 18, 1949 and February 3, 1950. A substantial part of the research presented here was carried out while the author was a Guggenheim fellow.