AN ISOPERIMETRIC PROBLEM FOR MULTIPLE INTEGRALS IN THE CALCULUS OF VARIATIONS

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1. Statement of the problem. The purpose of this paper is to find a set of necessary conditions which the solution of the following problem must satisfy. Consider a class of manifolds in (t,x)-space with equations in the form $x = x(t)$, where $x = (x^1, \cdots, x^m)$, $t = (t_1, \cdots, t_n)$, and t is in a closed connected region A which is homeomorphic to a closed n-cell, which satisfy the following conditions:

(a) The functions $x^i(t)$ $(i = 1, \cdots, m)$ are single-valued and continuous on the closed region A and have specified values on the entire boundary of A. The region A can be divided into a finite number of subregions on the interiors of which these functions are of class C″.

(b) The isoperimetric conditions

$$I^\beta(x) = \int_A f^\beta(t, x, p)\, dt = k_\beta \qquad (\beta = 1, \cdots, r),$$

where p denotes the matrix $(p_j^i = \partial x^i/\partial t_j)$ $(i = 1, \cdots, m; j = 1, \cdots, n)$ and the k's are constants, are satisfied.

(c) The equations

$$\phi^\gamma(t, x) = 0 \qquad (\gamma = 1, \cdots, s < m) \tag{1}$$

are satisfied everywhere on A. The problem is to find in this class of admissible manifolds one which minimizes the integral

$$I^0(x) = \int_A f^0(t, x, p)\, dt.$$

The solution of this problem will be denoted by $x_0(t)$.

It will be assumed that the functions $f^\alpha(t, x, p)$ $(\alpha = 0, 1, \cdots, r)$ and $\phi^\gamma(t, x)$ are continuous and have continuous partial derivatives of the first and second orders for t in the closed region A and for finite values of x and p. The matrix $(\phi^\gamma_{x^i}(t, x_0))$ will be assumed to have maximum rank everywhere on the closed region A.

An admissible *weak variation* η will be defined as a set of m functions $\eta^i(t)$ of class C″ on A which satisfy the equations,

$$\phi^\gamma_{x^i}(t, x_0)\eta^i = 0 \qquad (\gamma = 1, \cdots, s; i = 1, \cdots, m),$$

Received May 23, 1949; in revised form, August 15, 1949. The author wishes to thank Professors L. M. Graves and M. R. Hestenes for helpful suggestions during the preparation of this paper.