THE VARIATION OF SPECTRA

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I. **Introduction.** The question of whether or not the spectrum of an operator depends continuously on the operator arises naturally in consideration of integral and differential equations. For certain integral equations the problem has been solved by H. Weyl, R. Courant, and others [7; 1527]. F. Rellich [12] has considered self adjoint operators on Hilbert space. It is therefore natural to consider the corresponding question for operators on Banach spaces and for Banach algebras. Various conditions which imply the continuous variation of spectra are found.

The main theorems concern complex Banach algebras with unit. The function s taking an element x of the algebra into its spectrum $s(x)$ is in this case upper semi-continuous; this follows easily from the openness of the set of inverses in a Banach algebra. Consideration of the resolvent integral shows that full continuity, defined in terms of the Hausdorff distance between spectra, obtains at an element whose spectrum is totally disconnected. The ideal-theoretic methods of Gelfand [5; 8] show that in any case commutativity is sufficient to guarantee the continuity of s. The problem of continuity is also solved for the normal elements of any appropriate *-algebra. It is in fact shown that s is continuous on any set Φ having the following property, *to wit*, there exists a number $K > 0$ such that if $x \in \Phi$ and $(x + \lambda 1)^{-1}$ exists, then

$$| s[(x + \lambda 1)^{-1}] | > K \, || (x + \lambda 1)^{-1} ||.$$

(If E is a set of complex numbers, $| E | = \sup_{\lambda \in E} | \lambda |$, by definition.) A lemma of Arens [2; 272] is used to show that the set of normal elements has this property.

These theorems are generalized to Banach algebras without unit, and certain special results are obtained for algebras of linear operators.

A final generalization concerns closed operators on a Banach space. A definition of convergence is introduced, and the problem is then reduced to the consideration of linear operators.

II. **Preliminary remarks.** By *Banach Algebra* will be meant a complex Banach algebra with a unit 1 and satisfying the condition $|| xy || \leq || x || \, || y ||$, unless explicit modifications are made. The condition on the norm involves no real loss of generality, since any Banach algebra may be given an equivalent norm satisfying this relation [5; 3]. For any element x in a Banach algebra A, $r(x)$ is used to denote the set (the *resolvent* set) of complex numbers λ such that $x - \lambda 1$ has a 2-sided inverse in A. The *spectrum* of x, $s(x)$, is complement of $r(x)$ in C, the set of all complex numbers. It is well known that $s(x)$ is closed

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