ALMOST PERIODIC GEODESICS ON MANIFOLDS OF HYPERBOLIC TYPE

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1. **Introduction.** Morse, Hopf, Hedlund and others have studied the geodesics on certain Riemannian manifolds, defined by a metric and Fuchsian group in the unit circle, and related to a manifold of constant negative curvature. In this paper, based on the author's University of Virginia dissertation, the basic relations between the geodesics are extended to the n-dimensional case (§4) under more general conditions. Then the existence of almost periodic, non-periodic (called strictly almost periodic) geodesics is considered. The hypothesis of the principal theorem (Theorem 8.4) suggests the investigation of conditions under which strictly almost periodic geodesics exist on the companion hyperbolic manifold. In §9 symbolic sequences are used to guarantee their existence in the two-dimensional case when the manifold bears intersecting non-orthogonal periodic geodesics.

2. **The manifold** $M(f)$. For each integer $n \geq 2$ let S_n^* be the unit sphere in Euclidean n-space and let S_n denote the interior of S_n^*. Consider the n-dimensional quadratic form

$$(1) \qquad (ds)^2 = \frac{4f^2(x)\, dx_i\, dx_i}{(1 - x_i x_i)^2},$$

wherein $f(x) \equiv f(x_1, x_2, \cdots, x_n)$ is of class C^3 in S_n and such that for fixed constants a and b, $0 < a \leq f(x) \leq b$, for all $x = (x_1, x_2, \cdots, x_n) \in S_n$. The Riemannian manifold with points of S_n and fundamental form (1) will be denoted by $M(f)$. The length $L(C)$ assigned to each rectifiable curve by (1) will be called the D-length of C.

Using a theorem of McShane's [11; 210] on absolute minima, one is able to show that if z_1 and z_2 are points of S_n, there exists a geodesic segment joining z_1 and z_2 having D-length as small as any other continuous rectifiable curve segment joining z_1 and z_2. (See [6] for the proof, using Hilbert's Theorem for absolute minima, in the case $n = 2$.)

A geodesic segment g joining two points z_1 and z_2 of $M(f)$ is said to be of *class* A if $L(g)$ is the absolute minimum of the lengths of all geodesic segments of $M(f)$ between z_1 and z_2. A geodesic is said to be of class A if each segment is of class A. A class A geodesic is a simple curve and two class A geodesics can intersect at most once in S_n [13; Theorem 3]. If g is a class A geodesic segment joining z_1 and z_2, we define $D(z_1, z_2) = L(g)$. $D(z_1, z_2)$, called the D-distance between z_1 and z_2, provides a metric in S_n.

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