SOME GENERAL INVERSION FORMULAE FOR ANALYTIC FUNCTIONS

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1. Introduction. The problem of the inversion of a regular function in and on a closed contour finds its solution in the well-known Bürmann-Lagrange [3; 138] series (I). Its existence theorem [1; 123–125] may be stated as follows: If $f(z)$ is regular in a neighborhood of $z = z_0$, if $f(z_0) = w_0$ and $f'(z_0) \neq 0$ then the equation $f(z) = w$ has a unique solution, regular in the vicinity of w_0, of the form

$$\text{(I)} \qquad z = g(w) = z_0 + \sum_{r=1}^{\infty} a_r \frac{(w - w_0)^r}{r!},$$

where

$$a_r = [D_z^{r-1}\phi(z)]_{z_0}, \qquad \phi(z) = \frac{z - z_0}{f(z) - w_0}.$$

Three added expressions will be given for the coefficients a_r of the inverted series (I). Formula (II) is due to Ward [8] and is a slight modification, in symbols, of the original. Identity (A), for $z = z_0$, is a more explicit formulation of a_r since, as will be shown, it is the expansion of the determinant in (II), and so lends itself more readily to the formulation of the general term of the reversion series of any regular function. Further, because of this explicit character, numeric determinations of the coefficients are readily effected.

Formula (B) appears to be of greater theoretic implication. In any event it will be used to prove the formal validity of (A) and to set up connections between the different expressions for a_r.

It is assumed, in what follows, that f is a regular function on a region R in the complex plane, that for any z in R, $w = f(z)$ and $f'(z) \neq 0$. The existence of the inverse function g is supposed, regular on the image region R^* of R and for any w in R^*, $z = g(w) = g(f(z))$. In order to facilitate the consideration of the formal aspects of formulas (A) and (B), the analytic character of these expressions will be developed later (§4).

If $w = f(z)$ and $z = g(w) = g(f(z))$, the following symbolism is used for the differential operators:

$$w_i = f_i(z) \equiv \frac{d^i f(z)}{dz^i}, \qquad z_i = g_i(w) = g_i(f(z)) \equiv \frac{d^i g(w)}{dw^i} \qquad (i = 1, 2, \cdots).$$

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