A "SIMPSON'S RULE" FOR THE NUMERICAL EVALUATION OF WIENER'S INTEGRALS IN FUNCTION SPACE

By R. H. Cameron

Introduction. It is the purpose of this paper to make an attack on the rather formidable problem of the numerical evaluation of the Wiener integrals [7], [6], [2] of smooth functionals. We shall give two main formulas which we call our "rectangle rule" and our "Simpson's rule". The "rectangle rule" is the most simple and natural formula and expresses the Wiener integral as the limit of an n-fold Riemann integral. However, the limit is not approached very rapidly, so that even for a simple functional like

$$\left[\int_0^1 [x(s)]^2\, ds\right]^2 \tag{0.1}$$

the error obtained by using the n-fold integral in place of its limit is $O(n^{-1})$, not even $o(n^{-1})$. This means that to increase the accuracy by one decimal place we must multiply n (the multiplicity of the Riemann integral) by 10, thus taking it out of the range of possible computation by the fastest electronic computing machines which are contemplated even in the distant future.

On the other hand, by modifying the integral of the "rectangle rule" so as to include one extra variable, we obtain an $(n + 1)$-fold integral which approaches its limit faster than before. We call this our "Simpson's rule", both because of the rapidity with which it approaches its limit and because it gives the *exact* answer for all finite n when our functional is a "third degree polynomial functional". We shall show that the error obtained by using the Simpson's rule without taking the limit is only $O(n^{-2})$ (we actually give a definite estimate for the error, not just an order relation), and if the rule is applied to the example mentioned before, it does better than the estimate and gives an error which is $O(n^{-3})$ with a fairly small constant multiplying the n^{-3}. Extension of the methods of this paper should give accuracy to higher powers of n^{-1}. It is hoped that the greater accuracy of this Simpson's rule and its extensions and the consequent reduction in the number of integrations needed to achieve a given accuracy may bring the computation of the Wiener integrals of simple smooth functionals within the range of the high speed electronic computing machines of the not-too-distant future. If machine calculation were to be used, the functional to be integrated would have to be given by a comparatively simple explicit formula so that the machine could calculate each value of the functional as it went along and add it to the sum previously obtained without "remembering" each individual value. This would involve an essentially repetitive

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