PROJECTIVITIES IN RELATIVELY COMPLEMENTED LATTICES

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1. **Introduction.** It is well known that every complemented modular lattice satisfying the ascending chain condition is the direct union of a finite number of simple complemented modular lattices. Furthermore, the simple components are characterized completely by the fact that every two prime quotients are projective. Recently [2] this result has been extended to the case of arbitrary relatively complemented lattices satisfying the ascending chain condition.

Suppose we consider an arbitrary relatively complemented lattice of finite dimension. From the above remarks, this lattice will be simple if and only if for every two elements p and q covering the null element, the two quotients p/z and q/z are projective. In the complemented modular case, the simple lattices will correspond to projective spaces in which lines have at least three points; and hence the required projectivity can always be accomplished in two transposes. On the other hand, it is easy to give examples of simple, non-modular relatively complemented lattices in which two transposes are not sufficient to establish the desired projectivities.

The principal result of this paper is given in Theorem 3.1, which states in part:

Let L be a simple relatively complemented lattice of dimension $n > 1$. Then for any two points p and q, the projectivity between the quotients p/z and q/z can be accomplished using not more than $2[\frac{1}{2}(n + 1)]$ transposes.

(By the dimension of a lattice we mean the upper bound of the lengths of all complete chains from u to z.)

In proving Theorem 3.1, several lemmas are developed, some of which, particularly Lemmas 3.5 and 3.6, are interesting in their own right. They give some insight into the structure of relatively complemented lattices and how this structure differs from the special complemented modular lattices.

In §4 we shall give examples of simple relatively complemented lattices of dimension n in which there are two points p and q such that the projectivity between p/z and q/z actually requires $2\ [\frac{1}{2}(n + 1)]$ transposes. In §5 special results are given for relatively complemented semi-modular lattices.

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2. **Notation and Definitions.** Proper inclusion will be denoted by $a \supset b$, while a covers b will be indicated by $a \succ b$. With these exceptions, we shall follow in general the notation and terminology of [1]. The statement $a/b\ T\ c/d$ shall mean that the quotients a/b and c/d are transposes. $a/b\ P\ c/d$ will denote a/b projective to c/d, while $a/b\ P_m\ c/d$ will mean that the projectivity can be accomplished in not more than m transposes.

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