A SIMPLIFIED PROOF OF THE EXPANSION THEOREM FOR SINGULAR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

By Norman Levinson

1. We begin by a brief sketch of the approach we shall make to the problem. Consider the equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (\lambda - q(x))y = 0, \tag{1.0}$$

where $p(x)$, $q(x)$ and $p'(x)$ are real valued and continuous over the interval $0 \leq x < \infty$ and $p(x) > 0$. Now consider a boundary value problem with let us say $y(0) = 0$ and $y(b) = 0$ for some large $b > 0$. Let $y = \phi(x, \lambda)$ be the solution of (1.0) with $\phi(0, \lambda) = 0, \phi'(0, \lambda) = 1$. By the classical Sturm-Liouville theorem we have an increasing sequence λ_{bn} , $n = 0, 1, 2, \cdots$, of characteristic values such that $\phi(x, \lambda_{bn})$ are the corresponding characteristic functions. Let r_{bn} be the normalizing factor of $\phi(x, \lambda_{bn})$. Let the function $f(x) \varepsilon L^2(0, A)$ and $f(x) \equiv 0, x > A$. Then if $b > A$ we have the completeness relation for the Sturm-Liouville functions over $(0, b)$

$$\int_0^\infty | f(x) |^2 dx = \sum_{n=0}^\infty | \int_0^\infty f(x)\phi(x, \lambda_{bn}) dx |^2 r_{bn}^2 . \tag{1.1}$$

We define $\rho_b(u)$ as a monotone non-decreasing step-function which increases by r_{bn}^2 when u passes through λ_{bn} , is otherwise constant, and let us say at a point of discontinuity is defined by $\rho_b(\lambda_{bn}) = \rho_b(\lambda_{bm} - 0)$. Moreover $\rho_b(0) = 0$. Then (1.1) becomes

$$\int_0^\infty | f(x) |^2 dx = \int_{-\infty}^\infty | g(u) |^2 d\rho_b (u), \tag{1.2}$$

where

$$g(u) = \int_0^\infty f(x)\phi(x, u) dx.$$

Now if as $b \rightarrow \infty$ we can show $\rho_b(u)$ tends to a limit $\rho(u)$, if $| \rho_b(u) |$ is uniformly dominated for large $| u |$, and if the right side of (1.2) converges uniformly with respect to b then we shall get from (1.2)

$$\int_0^\infty | f(x) |^2 dx = \int_{-\infty}^\infty | g(u) |^2 d\rho (u). \tag{1.3}$$

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