SYMMETRIZABLE COMPLETELY CONTINUOUS LINEAR TRANSFORMATIONS IN HILBERT SPACE

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1. **Introduction.** By $\mathfrak{H}$ we shall denote a (generalized) abstract Hilbert space (that is, $\mathfrak{H}$ is a linear space of elements $x, y, z, \cdots$ for which there are defined the operations of addition $x + y$ and multiplication ax by complex numbers which obey the ordinary rules of vector algebra together with a complex-valued inner product (x, y) such that $(ax, y) = a(x, y)$, $(x + z, y) = (x, y) + (z, y)$, $(x, y) = \overline{(y, x)}$, and $(x, x) \geq 0$ with the equality sign holding if and only if x is the zero element θ of H; moreover, as a metric space with distance function $\rho(x, y) = (x - y, x - y)^{\frac{1}{2}} = | x - y |$, the space $\mathfrak{H}$ is complete), which is not assumed to be separable. This paper treats the spectral theory of a linear transformation K on $\mathfrak{H}$ to $\mathfrak{H}$ that is completely continuous and left-symmetrizable; that is, there is a symmetric bounded linear transformation S on $\mathfrak{H}$ to $\mathfrak{H}$ such that SK is symmetric. (A linear transformation T on $\mathfrak{H}$ to $\mathfrak{H}$ is *completely continuous* if it transforms the unit sphere $[x; | x | \leq 1]$ into a compact subset of $\mathfrak{H}$; an alternate definition is that if $\{x_n\}$ is a bounded sequence of points in $\mathfrak{H}$ then $\{Tx_n\}$ contains a convergent subsequence. A completely continuous transformation T is obviously bounded, that is, $\| T \| \equiv$ l.u.b. $| Tx |$ on $[x; | x | \leq 1]$ is finite. If T is a bounded linear transformation on $\mathfrak{H}$ to $\mathfrak{H}$, and $(Tx, y) \equiv (x, Ty)$, we shall say that T is *symmetric*. It is to be remarked that in the terminology most widely used in Hilbert space theory, such a transformation T is a "symmetric transformation with domain $\mathfrak{H}$" or a "self-adjoint transformation with domain $\mathfrak{H}$"; throughout this paper only transformations having domain $\mathfrak{H}$ will be considered. For a symmetric transformation (Tx, x) is real, and such a transformation is termed non-negative if $(Tx, x) \geq 0$ for every x in $\mathfrak{H}$. For brevity, we write $T \geq 0$ for "T is symmetric and non-negative." In particular, for a symmetric transformation, $T = T^*$, where for an arbitrary bounded linear transformation T, the adjoint T^* is determined by the definitive identity $(Tx, y) \equiv (x, T^*y)$; moreover, if T is symmetric, $\| T \| =$ l.u.b. $| (x, Tx) |$ on $[x; | x | \leq 1]$.) For brevity, *we shall use "transformation" to mean "bounded linear transformation on $\mathfrak{H}$ to $\mathfrak{H}$" and "symmetrizable" to mean "left-symmetrizable."*

An inequality on symmetrizable transformations that is basic for the discussion of this paper is proved in §2, while certain properties of completely continuous transformations are listed in §3. By means of an extremizing process that is a direct extension of one frequently used for symmetric transformations, it is shown in §4 that if K is completely continuous and symmetrizable by a non-negative transformation S then K has a real proper value in case $SK \neq 0$.

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