THE GENERALIZATION OF THE VALUATION THEORY

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1. Introduction. The two types of real field, the field of all real numbers and the p-adic number fields, have evoked an attempt to give a unified discussion of them. This is due to J. Kürschák [13] who has considered fields in which to every element a is assigned a positive real number $\phi(a)$ called the (absolute) value of a, so that $\phi(a)$ has the properties of the common absolute value:

$$\phi(a) = 0 \quad \text{if and only if} \quad a = 0; \tag{1}$$

$$\phi(ab) = \phi(a)\phi(b); \tag{2}$$

$$\phi(a + b) \leq \phi(a) + \phi(b). \tag{3}$$

Later A. Ostrowski [15] examined systematically the solutions of the functional equation $\phi(xy) = \phi(x)\phi(y)$ and determined all possible valuations of the rational numbers. He found that there are only two types of valuation, one equivalent to the common absolute value and one equivalent to the p-adic valuation. The main difference between them lies in the fact that in case of p-adic valuations, instead of the cited triangular inequality (3), a much stronger one holds, namely,

$$\phi(a + b) \leq \max \{\phi(a), \phi(b)\}. \tag{3'}$$

W. Krull [10], [11] pointed out that in case of *Archimedean* valuations (3), one needs two operations among the values, namely addition and multiplication, while in the *non-Archimedean* case (3′), only one operation is necessary. This remark admitted a new form of valuations in the non-Archimedean case, called exponential valuation which is got by putting $v(a) = -\log \phi(a)$. Then (2) and (3′) become

$$v(ab) = v(a) + v(b); \qquad v(a + b) \geq \min (v(a), v(b)).$$

Now it is allowed the exponents $v(a)$ to be elements of arbitrary linearly ordered Abelian groups. Krull [11] developed the general theory of such valuations and Schilling [16] extended it to the non-commutative case. The present paper shows how a further generalization can be carried out in the commutative case by means of partially ordered Abelian groups.

2 Partially ordered Abelian groups. Let Γ be an *additive Abelian group* consisting of elements denoted by Greek letters $\alpha, \beta, \gamma, \cdots$. We assume that an ordering relation $>$ is defined for some pairs of elements in Γ with the following axioms. (For another set of axioms, stated in the non-commutative case, see [6].)

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