RECURRENT DETERMINANTS OF LEGENDRE AND OF ULTRASPHERICAL POLYNOMIALS

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1. **Introduction.** For Legendre polynomials $P_n(x)$, P. Turán has recently established the inequality

$$(1) \qquad \Delta_n(x) \equiv P_{n-1}(x)P_{n+1}(x) - [P_n(x)]^2 \leq 0 \qquad (n \geq 1, -1 \leq x \leq 1)$$

with equality only for $x = \pm 1$. Turán's proof and three additional proofs of (1) were given by Szegö [18], who also extended the result to ultraspherical, Laguerre, and Hermite polynomials. One of the authors of the present paper gave sharper results concerning $\Delta_n(x)$ and concerning the analogous expression for ultraspherical polynomials and extended these results to Bessel functions [16]. In a forthcoming paper [2] Forsythe studies expressions of the form $P_n(x)P_{n+h+k}(x) - P_{n+h}(x)P_{n+k}(x)$.

The present paper will be concerned with the study of determinants of recurrent type. (These are sometimes also called persymmetric or Hankel determinants.) Being given a sequence of functions $a_0(x)$, $a_1(x)$, $a_2(x)$, $\cdots$, we shall consider the $(k - i + 1)$-rowed recurrent determinants $D[a(x)]_i^k \equiv | a_{i+r+s} |$, $r, s = 0, \cdots, k - i$, $0 \leq i \leq k$, and their minors. When no confusion will arise, the independent variable may be suppressed. In this paper we shall restrict ourselves to the case $a_j(x) \equiv P_j^{(\lambda)}(x)/P_j^{(\lambda)}(1)$, where $P_j^{(\lambda)}(x)$ denotes the ultraspherical polynomial of degree j and parameter λ. Our results become particularly simple for the case $\lambda = 1$ (Chebyshev polynomials). The Legendre polynomials $P_n(x)$ are the special case $\lambda = \frac{1}{2}$. Geronimus [4] has evaluated the determinants $D[P(x)]_0^n$ and $D[P(x)]_1^n$ and obtained particularly simple and elegant expressions for them. We shall derive his formulas by an entirely different method. The same formula for $D[P(x)]_0^n$ was discovered, but never published, by Szegö, who used yet another method.

The expression $\Delta_n(x)$ in (1) is the special case $D[P(x)]_{n-1}^n$. For $\Delta_n(x)$ we shall establish a simple identity, (2) below, from which (1) and certain other results follow. Some of these results, which involve the establishment of convexity properties, apply equally well to Legendre functions of the second kind and to Legendre functions of non-integral order.

2. **A differential identity.** The following result [1], [7] is fundamental in the present development.

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