A THEOREM ON QUARTIC POLYNOMIALS

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1. In this note, we are concerned with necessary and sufficient conditions which must be satisfied by the coefficients of a real quartic polynomial $f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ in order that $f(x) > 0$ if $x \geq 0$. We must have $a_0 > 0$, $a_4 > 0$. Write $x = ct$ where c is the positive number which satisfies $a_0c^4 = a_4$. It suffices to consider the quartic $f(ct)/a_4$ for $t > 0$. This quartic, however, has its first and last coefficients equal to 1. We may therefore confine our attention to quartics

$$f(x) = x^4 + ax^3 + bx^2 + cx + 1 \qquad (x > 0). \tag{1}$$

We define $M(b) = b^2 + 20b - 28 - (b - 6)\,|\,b - 6\,|$, so that $M(b) = 32(b - 2)$ if $b \geq 6$, $= 2(b + 2)^2$ if $b < 6$, and $M(b) \geq 0$ for all b. It will appear that if $M(b) \geq 8ac$, then $d = 12 + b^2 - 3ac \geq 0$. We can then define a function n by the equation $3n = 6 - b + d^{\frac{1}{2}}$. We shall see that $b + 2n - 2 \geq 4$. Write $L = L(b) = (n - 4)(b + 2n - 2)^{\frac{1}{2}} - (a + c)$. The complete solution of our problem is given by the

Theorem. *The polynomial* (1) *satisfies the condition*

$$f(x) > 0 \qquad (x > 0) \tag{2}$$

in one of the following mutually exclusive cases, and only in such cases:

(i) $a > 0, c > 0, b + 2 + 2(ac)^{\frac{1}{2}} > 0$;

(ii) $a > 0, c > 0, b + 2 + 2(ac)^{\frac{1}{2}} \leq 0, L < 0$;

(iii) $a < 0, c < 0, L < 0, M(b) > 8ac, b + 2 > 0$;

(iv) $ac \leq 0, L < 0$.

2. Let S denote the set of points whose rectangular co-ordinates (a, b, c) are such that the quartic (1) satisfies the condition (2). Let F denote the frontier of S. Given a, c, there is a uniquely determined number b_0 such that, if $b > b_0$, then (a, b, c) is a point of S; if $b < b_0$, then the corresponding $f(x)$ takes negative values for some $x > 0$; and if $b = b_0$, then the corresponding $f(x)$ is non-negative for $x > 0$ and has a zero of even multiplicity for a positive value of x, say $x = t$. The number b_0 is the unique number with the property that (a, b_0, c) is a point of F. The corresponding $f(x)$ is of the form $(x - t)^2[(x - t^{-1})^2 + sx]$. The last factor must be non-negative for $x > 0$. Hence we must have $s \geq 0$. Expanding the product and comparing coefficients, we find that

$$a = s - 2\left(t + \frac{1}{t}\right), \quad c = st^2 - 2\left(t + \frac{1}{t}\right), \quad b_0 = 2 - 2st + \left(t + \frac{1}{t}\right)^2. \tag{3}$$

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